Exercise (1). Suppose that the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Define the function $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(s, t) = \psi(s^2 t, s)$ for $(s, t) \in \mathbb{R}^2$. Find $\partial g / \partial s(s, t)$ and $\partial g / \partial t(s, t)$.

Solution (1). Define $f(s, t) = s^2 t$ and $h(s) = s$, which are polynomials in $s$ and $t$, and hence continuously differentiable. Then $g(s, t) = \psi(f(s, t), h(s))$, and by the Chain Rule,

$$\frac{\partial g}{\partial s}(s, t) = \frac{\partial \psi}{\partial f}(f(s, t), h(s)) \frac{\partial f}{\partial s}(s, t) + \frac{\partial \psi}{\partial h}(f(s, t), h(s)) \frac{\partial h}{\partial s}(s) = 2st \frac{\partial \psi}{\partial (s^2 t)}(s^2 t, s) + \frac{\partial \psi}{\partial s}(s^2 t, s),$$

and

$$\frac{\partial g}{\partial t}(s, t) = \frac{\partial \psi}{\partial f}(f(s, t), h(s)) \frac{\partial f}{\partial t}(s, t) + \frac{\partial \psi}{\partial h}(f(s, t), h(s)) \frac{\partial h}{\partial t}(s) = s^2 \frac{\partial \psi}{\partial f}(f(s, t), h(s)) + 0 = s^2 \frac{\partial \psi}{\partial (s^2 t)}(s^2 t, s).$$

Exercise (8). Suppose that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ have continuous second-order partial derivatives. Also suppose that there is a number $\lambda$ such that $f''(x) = \lambda f(x)$ and $g''(x) = \lambda g(x)$ for all $x \in \mathbb{R}$. Define the function $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x, y) = f(x)g(y)$ for $(x, y) \in \mathbb{R}^2$. Prove that $\frac{\partial^2 u}{\partial x^2}(x, y) - \frac{\partial^2 u}{\partial y^2}(x, y) = 0$ for every $(x, y) \in \mathbb{R}^2$.

Solution (8). Taking first-order partials we get

$$u_x(x, y) = f_x(x)g(y), u_y(x, y) = f(x)g_y(y)$$

and taking second-order partials we get

$$u_{xx}(x, y) = f_{xx}(x)g(y), u_{yy}(x, y) = f(x)g_{yy}(y)$$
and we know that $f_{xx}(x) = \lambda f(x)$ and $g_{yy}(y) = \lambda g(y)$. Then

$$u_{xx}(x, y) - u_{yy}(x, y) = f_{xx}(x)g(y) - f(x)g_{yy}(y) = \lambda f(x)g(y) - \lambda f(x)g(y) = 0$$

as desired.

**Exercise (11).** Suppose that the function $g : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable. For points $x$ and $p \in \mathbb{R}^n$, the Directional Derivative Theorem asserts that if $\psi(t) = g(x + tp)$ for $t \in \mathbb{R}$, then $\psi'(t) = (\nabla g(x + tp), p)$ for every $t \in \mathbb{R}$. Show that this formula is a special case of the Chain Rule.

**Solution (11).** Define $F : \mathbb{R} \to \mathbb{R}^n$ by $F(t) = x + tp$, which is continuously differentiable, so we can use the Chain Rule on $\psi(t) = g(F(t))$ to get

$$\psi'(t) = \frac{d}{dt} g(F(t)) = \sum_{i=1}^{n} \frac{\partial g}{\partial f_i}(F(t)) \frac{\partial F}{\partial t}(t) = \sum_{i=1}^{n} \frac{\partial g}{\partial f_i}(F(t))p_i = (\nabla g(F(t)), p) = (\nabla g(x + tp), p)$$

as desired.