Exercise (3). In example 18.8, find the exact values of $U(f, P_k)$ and $L(f, P_k)$.

Solution (3). The function in question is

$$f(x, y) = \begin{cases} 1 & y > x \\ 0 & y \leq x \end{cases}$$

for $(x, y) \in I$. $P_k$ is defined as chopping $[0, 1] \times [0, 1]$ into $k$ intervals of length $1/k$. Notice that $P_k$ will consist of $k^2$ squares of size $1/k^2$. Also observe that $f$ is constant on many of these squares, so $\inf f = \sup f$ on these squares. Exactly how many squares is $f$ constant on? Well, $f$ will not be constant on a square with the line $y = x$ passing through, which only happens on the squares $[i/k, (i + 1)/k] \times [i/k, (i + 1)/k]$ for $i \in \{0, \ldots, k - 1\}$, otherwise $y > x$ or $x > y$ on the whole square. By counting the $i$ we see that there are $k$ such squares. That leaves us with $k^2 - k = k(k - 1)$ (an even number of) squares with $f$ constant. Well, the line $y = x$ divides $I$ into two equal sized pieces and the partition $P_k$ chops $I$ into equal pieces, so there as many pieces above $y = x$ as below, hence there are $k^2 - k$ squares where $f = 1$ and $k^2 - k$ squares where $f = 0$. Then $\sup f = \inf f = 1$ on $k^2 - k$ squares, $\sup f = \inf f = 0$ on $k^2 - k$ squares, and $\sup f = 1, \inf f = 0$ on $k$ squares. Therefore,

$$U(f, P_k) = \sum_{i=1}^{k} \frac{1}{k^2} + \sum_{i=1}^{\frac{k^2 - k}{2}} 0 \cdot \frac{1}{k^2} + \sum_{i=1}^{\frac{k^2 - k}{2}} 1 \cdot \frac{1}{k^2} = \frac{k + \frac{k^2 - k}{2}}{k^2} = \frac{k + 1}{2k}$$

$$L(f, P_k) = \sum_{i=1}^{k} 0 \cdot \frac{1}{k^2} + \sum_{i=1}^{\frac{k^2 - k}{2}} 0 \cdot \frac{1}{k^2} + \sum_{i=1}^{\frac{k^2 - k}{2}} 1 \cdot \frac{1}{k^2} = \frac{k - 1}{2k}$$
Exercise (7). For the generalized rectangle $I = [0, 1] \times [0, 1] \in \mathbb{R}^2$, define

$$f(x, y) = \begin{cases} 
5 & \text{if } (x, y) \text{ is in } I \text{ and } x > 1/2 \\
1 & \text{if } (x, y) \text{ is in } I \text{ and } x \leq 1/2 
\end{cases}$$

Use the Archimedes-Riemann Theorem to show that the function $f : I \rightarrow \mathbb{R}$ is integrable.

Solution (7). We need to show that there is a sequence of partitions $P_k$ such that $\lim_{k \rightarrow \infty} U(f, P_k) - L(f, P_k) = 0$. Let $P_1 = \{[0, 1/2] \times [0, 1], [1/2, 1] \times [0, 1] \}$. We don’t really care about partitioning the y-axis at all. In general we let $P_k = \{[0, 1/2] \times [0, 1], [1/2, 1/(k+1) + 1/2] \times [0, 1], [1/(k+1) + 1/2, 1] \times [0, 1] \}$. Notice that the only rectangle that $f$ is non-constant on is $[1/2, 1/(k+1) + 1/2] \times [0, 1]$. Well, the volume of this rectangle is $\frac{1}{k+1}$. So, $\lim_{k \rightarrow \infty} U(f, P_k) - L(f, P_k) = \lim_{k \rightarrow \infty} \frac{1}{k+1} = 0$. By the Archimedes-Riemann Theorem, $f$ is integrable.

Exercise (9). Let $I$ be a generalized rectangle in $\mathbb{R}^2$ and suppose that the function $f : I \rightarrow \mathbb{R}$ assumes 0 except at a single point $x \in I$. Show that $f$ is integrable, then show $\int_I f = 0$. Is the same true for a generalized rectangle in $\mathbb{R}^n$?

Solution (9). Why don’t we use the equivalent condition for integrability from Theorem 18.10. Let $\epsilon > 0$. Write $I$ as $[a_1, b_1] \times [a_2, b_2]$. We also know $f(x) = c$ for some $c \in \mathbb{R}$. We build a partition $P_\epsilon$. In the x-direction, we partition $[a_1, b_1]$ by the points $\{a_1, x_1 - \sqrt{\epsilon}/(2\sqrt{c}), x_1 + \sqrt{\epsilon}/(2\sqrt{c}), b_1\}$. We do the same thing for the y-direction: $\{a_2, x_2 - \sqrt{\epsilon}/(2\sqrt{c}), x_2 + \sqrt{\epsilon}/(2\sqrt{c}), b_2\}$. Then the only rectangle where $f$ is non-constant is $[x_1 - \sqrt{\epsilon}/(2\sqrt{c}), x_1 + \sqrt{\epsilon}/(2\sqrt{c})] \times [x_2 - \sqrt{\epsilon}/(2\sqrt{c}), x_2 + \sqrt{\epsilon}/(2\sqrt{c}), b_2]$.

On this rectangle, $\sup f = c, \inf f = 0$, so then $U(f, P_\epsilon) - L(f, P_\epsilon) = c \cdot \epsilon/c = \epsilon$, which by Theorem 18.10 shows integrability. It is obvious that $\int_I f = \lim_{\epsilon \rightarrow 0} L(f, P_\epsilon) = 0$. We can do the same thing in $\mathbb{R}^n$ by replacing $\sqrt{\epsilon}/(2\sqrt{c})$ with $\sqrt{\epsilon}/(2\sqrt{c})$, and we get $\sup f = c, \inf f = 0$ on the middle rectangle and the difference between $U$ and $L$ will again be $\epsilon$. The integral will still be 0 since $L(f, P_\epsilon) = 0$ for all $\epsilon > 0$.

Exercise (14). Let $I$ be a generalized rectangle in $\mathbb{R}^n$ and suppose that the function $f : I \rightarrow \mathbb{R}$ is integrable. Let the number $M$ have the property that $|f(x)| \leq M$ for all $x \in I$. Prove that $|\int_I f| \leq M \cdot \text{vol}(I)$.

Solution (14). Let $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}, g(x) = M$. We show $g(x)$ is integrable on $I$. Well, no matter how you partition $I$, $f$ will always be constant on each rectangle by definition. Then $\sup g = \inf g = M$ on each rectangle of the partition, and hence $U(g, P) - L(g, P) = 0$ for any partition $P$. We can take any sequence of partitions $P_k$, then, and obtain $\lim_{k \rightarrow \infty} U(g, P_k) - L(g, P_k) = 0$. If we let $P_k$ just be the interval $I$ for all $k$, we get $\int g(x) = \lim_{k \rightarrow \infty} U(g, P_k) = \lim_{k \rightarrow \infty} L(g, P_k) = M \cdot \text{vol}(I)$.
\[ \lim_{k \to \infty} M \cdot \text{vol}(I) = M \cdot \text{vol}(I) \] Ok that may have been trivial. Anyways, notice that we can break \( f \) into positive and negative parts:

\[
\begin{align*}
  f^+ &:= \begin{cases} 
    f(x) & \text{if } f(x) \geq 0 \\
    0 & \text{if } f(x) < 0
  \end{cases} \\
  f^- &:= \begin{cases} 
    f(x) & \text{if } f(x) < 0 \\
    0 & \text{if } f(x) \geq 0
  \end{cases}
\end{align*}
\]

and that for any \( x \), \( f(x) = f^+(x) + f^-(x) \), and \( |f(x)| = f^+(x) - f^-(x) \). By Theorem 18.12, \( \int_I f^+ \geq 0 \), and \( \int_I f^- \leq 0 \) since \( f^+(x) \geq 0, f^-(x) \leq 0 \forall x \in I \).

Then by Theorem 18.13 and the triangle inequality, \( |\int_I f| = |\int_I f^+ + \int_I f^-| \leq \int_I f^+ + |\int_I f^-| = \int_I f^+ - \int_I f^- = \int (f^+ - f^-) = \int |f| \). Applying Theorem 18.12 once more, we have that since \( |f(x)| \leq g(x) \),

\[
\left| \int_I f \right| \leq \int_I |f| \leq g(x) = M \cdot \text{vol}(I).
\]

as desired.