19.1: 2, 4, 7, 8

Exercise (2). For the following functions, evaluate \( \iint_I f \), where \( I = [0, 1] \times [0, 1] \):

(a) 
\[
f(x, y) = \begin{cases} 
1 - x - y & \text{if } x + y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

(b) 
\[
f(x, y) = \begin{cases} 
x^2 + y^2 & \text{if } x^2 + y^2 \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

(c) 
\[
f(x, y) = \begin{cases} 
x + y & \text{if } x^2 \leq y \leq 2x^2 \\
0 & \text{otherwise}
\end{cases}
\]

Solution (2). We use Theorem 19.3.

(a) 
\[
\int_0^1 \left[ \int_0^{1-x} 1 - x - y dy \right] dx = \int_0^1 \left[ y - xy - \frac{1}{2} y^2 \right]_0^{1-x} dx \\
= \int_0^1 (1 - x)(1 - x) - \frac{1}{2} (1 - x)^2 dx = \int_0^1 \frac{1}{2} (1 - x)^2 dx \\
= \left[ \frac{1}{2} x - \frac{1}{2} x^2 + \frac{1}{6} x^3 \right]_0^1 = \frac{1}{6}
\]
\[
\int_0^1 \left[ \int_0^{\sqrt{1-x^2}} x^2 + y^2 \, dy \right] \, dx = \int_0^1 \left( \frac{2}{3} x^2 + \frac{1}{3} \right) \sqrt{1-x^2} \, dx
\]

and let \( u = \sin x \), so \( du = \cos x \, dx \). Substituting, we have

\[
\frac{1}{3} \left[ \int_0^{\sin^{-1} 1} (2 \sin^2 u + 1) \cos^2 u \, du = \frac{1}{3} \int_0^{\pi/2} (1 - \cos 2u + 1) \frac{1 + \cos 2u}{2} \, du \right.
\]

\[
\frac{1}{3} \int_0^{\pi/2} \frac{(2 - \cos 2u)(1 + \cos 2u)}{2} \, du = \frac{1}{6} \int_0^{\pi/2} 2 + \cos 2u - \cos^2 2u \, du = \frac{\pi}{6} - \frac{1}{6} \int_0^{\pi/2} \cos^2 2u \, du
\]

\[
\frac{\pi}{6} - \frac{1}{3} \int_0^{\pi/2} \frac{1 + \cos 4u}{2} \, du = \frac{\pi}{6} - \frac{\pi}{24} = \frac{\pi}{8}
\]

Exercise (4). For a continuous function \( f : [a, b] \times [a, b] \to \mathbb{R} \), prove Dirichlet’s formula:

\[
\int_a^b \left[ \int_a^x f(x, y) \, dy \right] \, dx = \int_a^b \left[ \int_y^b f(x, y) \, dx \right] \, dy
\]

Solution (4). This follows from the observation that this integration is over the domain \( D = \{(x, y) : a \leq y \leq x \leq b\} \) from which, using Theorem 19.3 and the inequalities \( a \leq x \leq b, a \leq y \leq x \) to get the left hand side of the equality \( \int_a^b \left[ \int_a^x f(x, y) \, dy \right] \, dx \) and applying Theorem 19.3 again with the inequalities \( a \leq y \leq b, y \leq x \leq b \) to get the right hand side of the equality \( \int_a^b \left[ \int_y^b f(x, y) \, dx \right] \, dy \), as desired.

Exercise (7). Follow the proof of Theorem 19.3 and thereby provide a proof of Theorem 19.8

Solution (7). (Copy of the proof, with minor changes) The set \( D \) is a Jordan domain since its boundary consists of the union of \( 2n \) graphs, each of which is the graph of a continuous function on a bounded interval. Choose an interval \([a_n, b_n]\)
such that the rectangle $I = [a_1, b_1] \times \ldots \times [a_n, b_n]$ contains $D$ and let $\hat{f} : I \to \mathbb{R}$ be the zero extension of $f$ to $I$. Theorem 18.24 implies that $\hat{f}$ is integrable. Using lemma 18.21 in the case where $n = 1$, with $I = [a_{k+1}, b_{k+1}]$ and $J = [h(x), g(x)]$, we conclude that $A(x) \equiv \int_{a_n}^{b_n} \hat{f}(x, y)dy = \int_{h(x)}^{g(x)} f(x, y)dy$ for all $x \in [a_1, b_1] \times \ldots \times [a_{n-1}, b_{n-1}]$. The formula now follows from Fubini’s Theorem (19.7).

**Exercise (8).** Show that the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined in Example 19.5 is not integrable.

**Solution (8).** Suppose to the contrary that $f$ is integrable. By Theorem 18.11, the restriction of $f$ to $G = [0,1] \times [3/4, 1]$ is also integrable. Then we can find an Archimedean sequence of partitions $P_k$ such that $\lim_{k \to \infty} L(f, P_k) = \lim_{k \to \infty} U(f, P_k)$. In every subrectangle in $P_k$ there are points with irrational and rational $x$ values. Also notice that $y \geq 3/4$ for all $y$ in every subrectangle of $P_k$. Then for each $J_{ik} \in P_k, \sup_{(x,y) \in J_{ik}} f(x, y) \geq 3/2$, and for each $J_{ik} \in P_k, \inf_{(x,y) \in J_{ik}} f(x, y) = 1$. Then

$$\lim_{k \to \infty} L(f, P_k) = \lim_{k \to \infty} \sum_{J_{ik} \in P_k} \inf_{(x,y) \in J_{ik}} f(x, y) \text{vol}(J_{ik})$$

$$= \lim_{k \to \infty} \sum_{J_{ik} \in P_k} \text{vol}(J_{ik}) = \lim_{k \to \infty} \frac{1}{4} = \frac{1}{4}$$

and

$$\lim_{k \to \infty} U(f, P_k) = \lim_{k \to \infty} \sum_{J_{ik} \in P_k} \sup_{(x,y) \in J_{ik}} f(x, y) \text{vol}(J_{ik})$$

$$\geq \lim_{k \to \infty} \sum_{J_{ik} \in P_k} \frac{3}{2} \text{vol}(J_{ik}) = \lim_{k \to \infty} \frac{3}{8} = \frac{3}{8}$$

which gives us that

$$\lim_{k \to \infty} U(f, P_k) \geq \frac{3}{8} > \frac{1}{4} = \lim_{k \to \infty} L(f, P_k)$$

which contradicts $P_k$ being an Archimedean sequence of partitions. So, there does not exist an Archimedean sequence of partitions for $f$ restricted to $G$ so $f$ is not integrable over $G$ and by Lemma 18.11 $f$ is not integrable over $[0,1] \times [0,1]$. 
