Exercise (2). Show that the linear volume transformation formula (19.16) is a special case of formula (19.11).

Solution (2). First, we note that statement 19.16 is
\[ \text{vol}(\text{d}\Psi(x)(J)) = |\det D\Psi(x)|\text{vol}J \]
and statement 19.11 is
\[ \int_{\Phi(K)} f(x)dx = \int_K f(\Phi(u))|\det D\Phi(u)|du \]
where \( \Phi(u) = x \). Ok, now define \( \Phi(J) = d\Psi(x)(J) \) to be our smooth change of variables. Since \( \Psi \) is defined to be a linear map, it has constant derivative \( D\Psi = D\Psi(x) \) for all \( x \in O \) as defined on page 511. Then \( d\Psi(x)(J) = d\Psi(J) \) is also a linear map, and hence is 1-1 and invertible by the assumptions of (i) on page 511. Now, applying the change of variables theorem, we have
\[ \text{vol}(\text{d}\Psi(x)(J)) = \int_{d\Psi(x)(J)} 1d\text{d}(\Psi(x)(u)) = \int_J |\det D\Psi(x)(v)|dv = \int_J |\det D\Psi(x)|dv \]
where \( d\Psi(x)(J) \) is a generalized rectangle contained in \( O \).

Exercise (7). Mean Value Property for Integrals. Suppose that \( G : O \to \mathbb{R}^n \) is a smooth change of variables on the open subset \( O \subseteq \mathbb{R}^n \), \( h : G(O) \to \mathbb{R} \) is continuous, and \( J \) is a generalized rectangle contained in \( O \).

(a) Use the Extreme Value Theorem to choose points \( u, v \in J \) at which the function \( h \circ G : J \to \mathbb{R} \) assumes a minimum and a maximum, respectively, and then use the monotonicity property of the integral to show that
\[ h(G(u)) \leq \frac{1}{\text{vol}(G(J))} \int_{G(J)} h(x)dx \leq h(G(v)) \]
(b) Apply the Intermediate Value Theorem to the function $\alpha : [0, 1] \to \mathbb{R}$ defined by $\alpha(t) = h(G(tu + (1 - t)v))$, $0 \leq t \leq 1$ and part (a) to find a point $w$ on the segment between $u$ and $v$ at which

$$\int_{G(J)} h(x)dx = h(G(w)) \text{vol}(G(J))$$

(c) In the case where $G(x) = x$ for all $x \in \mathcal{O}$, obtain the following consequence:

$$\int_{J} h(x)dx = h(w) \text{vol}(J)$$

Solution (7). (a) First, notice that since $J$ is a generalized rectangle, it is sequentially compact in $\mathbb{R}^n$, and since $h$ is continuous, $h \circ G$ is continuous by Theorem 12.33. Then by Theorem 12.37 (Extreme Value Theorem), $h \circ G$ attains both a max and min on $J$, and we’ll call $u$ a minimizer and $v$ a maximizer. Thus we know that

$$h(G(u)) \leq h(G(x)) \leq h(G(v)) \forall G(x) \in G(J)$$

Denote $y = G(x)$. Then, by Theorem 18.13

$$\int_{G(J)} h(G(u))dy \leq \int_{G(J)} h(y)dy \leq \int_{G(J)} h(G(v))dy$$

$$h(G(u)) \int_{G(J)} 1dy \leq \int_{G(J)} h(y)dy \leq h(G(v)) \int_{G(J)} 1dy$$

$$h(G(u)) \text{vol}(G(J)) \leq \int_{G(J)} h(y)dy \leq h(G(v)) \text{vol}(G(J))$$

$$h(G(u)) \leq \frac{1}{\text{vol}(G(J))} \int_{G(J)} h(y)dy \leq h(G(v))$$

as desired.

(b) First, note that $J$ is convex since it is a generalized rectangle and so $f(t) = tu + (1 - t)v, t \in [0, 1]$ is contained in $J$. Also $f(t) = tu + (1 - t)v$ is a continuous function (it’s a line segment), and by Theorem 12.33 $\alpha : [0, 1] \to \mathbb{R}, \alpha(t) = h(G(f(t)) = h(G(tu + (1 - t)v))$ is continuous on $[0, 1]$. Now we have 2 cases:

(i) $\frac{1}{\text{vol}(G(J))} \int_{G(J)} h(y)dy = h(G(v))$ or $\frac{1}{\text{vol}(G(J))} \int_{G(J)} h(y)dy = h(G(u))$

(ii) $h(G(u)) < \frac{1}{\text{vol}(G(J))} \int_{G(J)} h(y)dy < h(G(v))$

The first case is obvious:

$$\int_{G(J)} h(y)dy = h(G(u)) \text{vol}(J)$$ or

$$\int_{G(J)} h(y)dy = h(G(v)) \text{vol}(J)$$
In the second case we can invoke the Intermediate Value Theorem, that is, we know that there exists \( t_0 \in [0, 1] \) such that
\[
\alpha(t_0) = h(G(t_0 u + (1 - t_0)v)) = \frac{1}{\text{vol}(G(J))} \int_{G(J)} h(y) dy
\]
so let \( w = ut_0 + (1 - t_0)v \), which gives us
\[
\int_{G(J)} h(y) dy = h(G(w)) \text{vol}(G(J))
\]
as desired.

(c) I’m not really sure why the book even states this as a problem. Obviously when \( G(x) = x, G(J) = J \) so just replacing \( G(J) \) with \( J \) and \( G(w) \) with \( w \) gives
\[
\int_{J} h(y) dy = h(w) \text{vol}(J)
\]

Exercise (11). Suppose that \( \Psi : \mathcal{O} \to \mathbb{R}^n \) is a smooth change of variables on the open subset \( \mathcal{O} \subseteq \mathbb{R}^n \) and that \( K \) is a closed bounded subset of \( \mathcal{O} \). Prove that there is a positive number \( c \) such that for any two points \( u, v \in K \)
\[
||\Psi(u) - \Psi(v)|| \geq c ||u - v||
\]
(Argue by contradiction using the Nonlinear Stability Theorem)

Solution (11). Suppose by way of contradiction that for all \( c > 0 \) there exist \( u, v \in K \) such that
\[
||\Psi(u) - \Psi(v)|| < c ||u - v||
\]
Now, let’s make two sequences \( \{u_k\}, \{v_k\} \subseteq K \) by picking the \( u_k, v_k \) that satisfy
\[
||\Psi(u_k) - \Psi(v_k)|| < \frac{1}{k} ||u_k - v_k||
\]
for each \( k \in \mathbb{N} \). Now, since \( K \) is compact, there exist converging subsequences \( \{u_{k_j}\}, \{v_{k_j}\} \) converging to \( u \) and \( v \) in \( K \), respectively. There are two cases:

1. \( u = v \)
2. \( u \neq v \)

First suppose that \( u = v \). By the Nonlinear Stability Theorem and \( D\Psi(u) \) invertible we get that there exists an \( \epsilon > 0 \) such that \( ||\Psi(x) - \Psi(y)|| \geq c ||x - y|| \) for some \( c > 0 \) and for all \( x, y \in \mathcal{B}_\epsilon(u) \). Also, by convergence, we can find \( M_1 \in \mathbb{N} \) such that for \( j \geq M_1, ||u_{k_j} - u|| < \epsilon, ||v_{k_j} - v|| < \epsilon \), so \( u_{k_j}, v_{k_j} \in \mathcal{B}_\epsilon(u) \), and hence
\[
||\Psi(u_{k_j}) - \Psi(v_{k_j})|| \geq c ||u_{k_j} - v_{k_j}||
\]
for all \( j \geq M_1 \). Now, by the Archimedean principle we can find \( M_2 \) such that \( \frac{1}{M_2} < c \). Choose \( M = \max(M_1, M_2) \). Then for \( j \geq M \), we have

\[
||\Psi(u_{k_j} - \Psi(v_{k_j})|| < \frac{1}{M}||u_{k_j} - v_{k_j}|| < c||u_{k_j} - v_{k_j}||
\]

but since \( u_{k_j}, v_{k_j} \in B_\epsilon(u) \),

\[
||\Psi(u_{k_j}) - \Psi(v_{k_j})|| \geq c||u_{k_j} - v_{k_j}||
\]

a contradiction. That leaves us with case 2, suppose \( u \neq v \). Then we have as \( \lim_{j \to \infty} \frac{1}{E_j}||u_{k_j} - v_{k_j}|| \leq \lim_{j \to \infty} \frac{1}{E_j}d = 0 \), so

\[
\lim_{j \to \infty} ||\Psi(u_{k_j}) - \Psi(v_{k_j})|| = 0
\]

Which means by continuity that \( \Psi(u) = \Psi(v) \). Then \( u \neq v \) and \( \Psi(u) = \Psi(v) \), which contradicts \( \Psi \) being injective.

Therefore, there must exist a \( c > 0 \) such that \( ||\Psi(u) - \Psi(v)|| \geq c||u - v|| \) for all \( u, v \in K \).