Exercise (4). Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(u) > 0$ if $u \in \mathbb{R}$ has at least one rational component. Prove $f(u) \geq 0$ for all points $u \in \mathbb{R}$.

Solution (4). For every $u \in \mathbb{R}^n$ with a rational component, we already have that $f(u) \geq 0$. So, we need only be concerned with elements with all irrational components. Let $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Fix $\epsilon > 0$. The continuity of $f$ implies that there exists $\delta > 0$ such that if $\text{dist}(u, v) < \delta$, $\text{dist}(f(u), f(v)) < \epsilon$. Since the rationals are dense in $\mathbb{R}$, we can find a rational number $v_1$ such that $u_1 < v_1 < u_1 + \delta$. Then $0 < v_1 - u_1 < \delta$. We create the vector $v \in \mathbb{R}^n$ identical to $u$ except with $v_1$ in the first component so that $\text{dist}(u, v) = \sqrt{(u_1 - v_1)^2 + \ldots + (u_n - v_n)} = \sqrt{(u_1 - v_1)^2 + 0 + \ldots + 0} = |u_1 - v_1| < \delta$. By continuity and $f(v) > 0$, we have that

$$|f(u) - f(v)| < \epsilon$$
$$f(u) - f(v) > -\epsilon$$
$$f(u) > f(v) - \epsilon > 0 - \epsilon = -\epsilon.$$ 

Since $\epsilon > 0$ can be chosen arbitrarily small, we have that $f(u) \geq 0$.

Exercise (6). Suppose that the function $f, g : \mathbb{R}^n \to \mathbb{R}$ are both continuous. Prove that the set $A = \{u \in \mathbb{R}^n | f(u) = g(u) = 0\}$ is closed in $\mathbb{R}^n$.

Solution (6). Let $\{u_k\}_{k \in \mathbb{N}}$ be a sequence contained in $A$ converging to $u \in \mathbb{R}^n$. We show that $u \in A$. Let $\epsilon > 0$. By continuity of $f$ and $g$, there exists $\delta_1, \delta_2$ such that $||u_k - u|| < \delta_1$ implies
\[ |f(u_k) - f(u)| < \epsilon, \text{ and } ||u_k - u|| < \delta_2 \text{ implies } |g(u_k) - g(u)| < \epsilon. \]  
Let \( \delta = \min(\delta_1, \delta_2) \). Since \( u_k \to u \), \( \exists K \in \mathbb{N} \) such that \( k \geq K \) guarantees \( ||u_k - u|| < \delta \). Then by continuity of \( f \) and \( g \), \( |f(u_k) - f(u)| < \epsilon \) and \( |g(u_k) - g(u)| < \epsilon \). Since \( u_k \in A \), we have \( f(u_k) = g(u_k) = 0 \) \( \forall k \). Then we have \( |f(u)|, |g(u)| < \epsilon \). Since \( \epsilon \) was arbitrary, \( f(u) = g(u) = 0 \), which completes the proof.

Alternatively, we can utilize Corollary 11.13 as follows: Notice that \( A = \{ u \in \mathbb{R}^n | g(u) \leq 0 \} \cap \{ u \in \mathbb{R}^n | g(u) \geq 0 \} \cap \{ u \in \mathbb{R}^n | f(u) \leq 0 \} \cap \{ u \in \mathbb{R}^n | f(u) \geq 0 \} \). Since \( f \) and \( g \) are continuous, these four sets satisfy the hypotheses of Corollary 11.13 and therefore must be closed. By Proposition 10.17, the intersection of these four sets is also closed and hence \( A \) is closed.

**Exercise (10).** Let \( O \) be an open subset of \( \mathbb{R}^n \) and suppose that the function \( f : O \to \mathbb{R} \) is continuous. Suppose that \( u \) is a point in \( O \) at which \( f(u) > 0 \). Prove that there is an open ball \( B \) about \( u \) such that \( f(v) > f(u)/2 \) for all \( v \in B \).

**Solution (10).** Choose \( \epsilon = f(v)/2 \). By the continuity of \( f \), we know that there exists \( \delta_1 > 0 \) such that \( ||u - v|| < \delta_1 \) implies \( |f(u) - f(v)| < \epsilon \). Since \( O \) is open, there exists an open ball of radius \( r \) about \( u \) contained in \( O \). Choose \( \delta = \min(\delta_1, r) \). We are guaranteed to have \( B_{\delta} \subseteq O \) since \( B_{\delta} \subseteq B_r \). Then by continuity of \( f \), if \( v \in B_{\delta}, |f(u) - f(v)| < \epsilon = f(u)/2 \), which implies \( f(v) > f(u) - f(u)/2 = f(v)/2 \). This concludes the proof.