Instructor: Todd Fisher

Math 342 Section 1: Winter 2012
Final Exam

PRINT NAME:_______________________________________

Sign the following pledge below:
I pledge on my honor that I have not given or received any unauthorized assistance on this examination.

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Answers to the questions should be written directly on the exam, and should be written neatly and correctly. Note that the first 6 questions are true-false. Mark T for true, F for false. These questions are worth 3 points each. The points for these remaining questions are indicated on the exam.

True-false questions

1. Let $D = \mathbb{Q} \cap [0,1]$. Then $D$ is a Jordan domain in $\mathbb{R}$.
   F

2. If $(X,d)$ is a metric space and $\{a_n\}$ is a Cauchy sequence in $X$, then $\{a_n\}$ converges.
   F

3. There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,\cdot)$ and $f(\cdot,y)$ are both integrable on $[0,1]$, but the iterated integrals are not equal.
   T

4. If $O \subset \mathbb{R}^n$ is open, $f,g : O \to \mathbb{R}$ are both continuously differentiable, and $S := \{x \in O : g(x) = 0\}$ contains a point $u$ that is an extreme point for $f : S \to \mathbb{R}$ where $\nabla g(u) \neq 0$, then there exists $\lambda \in \mathbb{R}$ where $\nabla f(u) = \lambda \nabla g(u)$.
   T

5. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(u) \geq \|u\|$ for every $u \in \mathbb{R}^n$. Then $f^{-1}([0,1])$ is sequentially compact.
   T

6. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$. If $g \circ f$ is differentiable at $x$, then $D(g \circ f)(x) = Dg(f(x))DF(x)$.
   T
7. (6 points) Define the terms in boldface by completing the sentence.

(a) For \( I \) a generalized rectangle and \( f : I \to \mathbb{R} \) a function. The **upper integral** of \( f \)

(b) A set \( D \subset \mathbb{R}^n \) has **Jordan content 0** if

8. (8 points) Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( F(x, y) = (2x \sin y, y \cos x) \).

   (i) Calculate \( DF|_{(0, \pi/2)} \).

   \[
   DF = \begin{bmatrix}
   2 \sin y & 2x \cos y \\
   -y \sin x & \cos x
   \end{bmatrix}
   \]

   Evaluated at \((0, \pi/2)\) we have

   \[
   DF|_{(0, \pi/2)} = \begin{bmatrix}
   2 & 0 \\
   0 & 1
   \end{bmatrix}
   \]

   (ii) Is \( F \) locally bijective in a neighborhood of \((0, \pi/2)\)? (Justify your answer).

   Yes since \( DF|_{(0, \pi/2)} \) is invertible we know from the Inverse Function Theorem that there exists a neighborhood \( U \) of \((0, \pi/2)\) such that \( F : U \to F(U) \) is a bijection.
9. (12 points) Let $K$ be a non-empty subset of $\mathbb{R}^n$. Define the characteristic function of $K$ by

$$
\chi_K(x) = \begin{cases} 
1 & x \in K \\
0 & x \notin K 
\end{cases}
$$

Show that $\chi_K$ is discontinuous at $x$ if and only if $x$ is a boundary point of $K$.

**Proof.** Let $x \in \text{int}(K)$ then there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset K$. So $\chi_K(y) = 1$ for all $y \in B_\epsilon(x)$ and $f$ is continuous at $x$.

A similar argument applies to points $x \in \mathbb{R}^n$ contained in the exterior of $K$.

For $x \in \partial K$ there exists sequences $x_j \to x$ and $y_j \to x$ such that $x_j \in K$ and $x_j \neq x$ for all $j$, and $y_j \notin K$ and $y_j \neq x$ for all $j$.

**Case 1:** Suppose that $x \in K$. Then $\lim_{j \to \infty} \chi_K(y_j) = 0 \neq \chi_K(x)$ and $\chi_K$ is discontinuous at $K$.

**Case 2:** Suppose that $x \notin K$. Then $\lim_{j \to \infty} \chi_K(x_j) = 1 \neq \chi_K(x)$ and $\chi_K$ is discontinuous at $K$. 
10. (12 points) Calculate
\[ \iint_{D} (y^2 - x^2)e^{(x^2+y^2)/2} \, dx \, dy \]
where \( D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x + y \leq 2, 0 \leq y - x \leq 2\} \). (Hint: Try the transformation \( x = u - v \) and \( y = u + v \).)

Notice the Jacobian of the transformation is \( |\det(A)| = 2 \) since
\[ A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \]

Then the integral transforms to
\[ \int_{0}^{1} \int_{0}^{1} 4uve^{u^2+v^2} 2 \, du \, dv = 2(e^2 + 1). \]
11. (12 points) Consider the system
\[ \begin{align*}
x^3 + x^2 y + \sin(x + y) &= 0 \\
\ln(1 + x^2) + 2x + (yz)^4 &= 0
\end{align*} \]

Notice that \((x, y, z) = (0, 0, 0)\) is a solution of this system. What does the implicit function theorem tell us about the solutions near \((1, 1, 1)\)?

There is a typo (sorry). It should read what does the implicit function theorem tell us about solutions near \((0, 0, 0)\). Notice that \(DF|_{(0,0,0)} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}\) where \(F(x, y, z) = (x^3 + x^2 y + \sin(x + y), \ln(1 + x^2) + 2x + (yz)^4)\). So the solutions to \(F(x, y, z) = (0, 0)\) near \((0,0,0)\) are a curve where \(x\) and \(y\) can be represented as functions of \(z\). So \(F(g(z), h(z), z) = (0, 0)\) for points near \((0,0,0)\).
12. (8 points) Compute the integral

\[ \int_{\Gamma} -ydx + xdy + dz. \]

along the curve \( \Gamma \) given by \( \gamma(t) = (\cos t, \sin t, t) \) and \( \gamma : [0, 2\pi] \to \mathbb{R}^3 \).

We have

\[ \int_{\Gamma} -ydx + xdy + dz = \int_0^{2\pi} -\sin t(-\sin t) dt + \cos t \cos t dt + dt = \int_0^{2\pi} 2dt = 4\pi. \]
13. (12 pts) Let $S$ be the disk $x^2 + z^2 \leq 1$ where $y = 0$. Compute $\int_{\partial S} \mathbf{F} \circ \mathbf{T} \, ds$ where

$$\mathbf{F}(x, y, z) = (xz + \sqrt{x^3 + x^2 + 2}, xy, xy + \sqrt{z^3 + z^2 + 2})$$

and assume $\partial S$ is oriented counterclockwise.

Again there is a typo. I’m sorry. Instead of $\mathbf{F} \circ \mathbf{T}$ it should be $\langle \mathbf{F}, \mathbf{T} \rangle$. One can try to compute this directly, but it is easier to apply Stokes’s Formula. In this case $\text{curl} \, \mathbf{F} = (x, x - y, y)$ and $\eta = (0, 1, 0)$. So $\langle \text{curl} \, \mathbf{F}, \eta \rangle = x - y$. Using the parameterization $\phi(r, \theta) = (r \cos \theta, 0, r \sin \theta)$ we see that $||\phi_r \times \phi_\theta|| = r$. Then $\int_{\partial S} \langle \mathbf{F}, \mathbf{T} \rangle \, ds = \int_0^{2\pi} \int_0^1 r \cos \theta \ r dr d\theta = 0$. 
14. (12 pts) State and prove the First Order Approximation Theorem.