Homework Assignment 10

March 7, 2014

1. Let $f : M \to \mathbb{R}$ be smooth. compute the coordinate representations for $df$.
   
   (a) $M = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and let $f(x, y) = \frac{x}{x^2 + y^2}$.
   
   (b) $M = \mathbb{R}^n$ and $f(x) = ||x||^2$ where the norm is the Euclidean norm.

2. A covariant $k$-tensor is alternating if
   
   $$T(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = -T(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k).$$
   
   Prove the following are equivalent.

   (a) $T$ is alternating
   
   (b) For all vectors $x_1, \ldots, x_k$ and permutations $\sigma \in S_k$ we have
       $$T(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = (\text{sgn}\sigma)T(x_1, \ldots, x_k)$$

   (c) $T = 0$ whenever two arguments are equal
   
   (d) $T(x_1, \ldots, x_k) = 0$ whenever $(x_1, \ldots, x_k)$ are linearly dependent.

   (e) With respect to any basis the components $T_{i_1 \ldots i_k}$ change sign whenever two indices are interchanged.

3. Let $T$ be a tensor prove that

   (a) $\text{Alt}(T)$ is alternating and
   
   (b) $T$ is alternating if and only if $T = \text{Alt}(T)$.

4. Compute the components $R^l_{ijk}$ of the curvature tensor and the sectional curvature for
(a) the sphere,
(b) the cylinder, and
(c) the hyperbolic upper half plane.