Homework Assignment 12

March 31, 2014

As a reminder if we let \( r(u, v) \) be a parameterized surface and
\[
\alpha(t) = r(u(t), v(t))
\]
where \( t \in (-\epsilon, \epsilon) \) a smooth parameterized curve with \( p = \alpha(0) = r(u_0, v_0) \) and \( \alpha'(0) = r_u u' + r_v v' \), then first fundamental form is
\[
I_p(\alpha'(0)) = E(u')^2 + 2Fu'v' + G(v')^2
\]
where \( E(u_0, v_0) = \langle r_u, r_u \rangle_p, \) \( F(u_0, v_0) = \langle r_u, r_v \rangle_p, \) and \( G(u_0, v_0) = \langle r_v, r_v \rangle_p. \)

For the derivative of the Gauss map we let \( dN(\alpha'(0)) = N(u(t), v(t)) = N_u u' + N_v v' \) where \( N_u, N_v \in T_pS \) defined by \( N_u = dN_p r_u \) and \( N_v = dN_p r_v. \)

The second fundamental form for a surface is related to the shape operator and given by
\[
II_p(\alpha'(0)) = -\langle dN(\alpha'(0)), \alpha'(0) \rangle
\]
\[
= -\langle N_u u' + N_v v', r_u u' + r_v v' \rangle = e(u')^2 + 2fu'v' + g(v')^2
\]
where \( e = -\langle N_u, r_u \rangle = \langle N, r_{uu} \rangle, \)
\[
f = -\langle N_v, r_u \rangle = \langle N, r_{uv} \rangle = \langle N, r_{vu} \rangle = -\langle N_u, r_v \rangle, \)
and \( g = -\langle N_v, r_v \rangle = \langle N, r_{vv} \rangle. \)

From the Weingarten equation we can derive that the Gaussian curvature (which is also the sectional curvature) is
\[
K = \frac{eg - f^2}{EG - F^2},
\]
and the mean curvature is
\[
H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}.
\]
Furthermore, if
\[ dN_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \]
then
\[ a_{11} = \frac{fF - eG}{EG - F^2}, \]
\[ a_{12} = \frac{gF - fG}{EG - F^2}, \]
\[ a_{21} = \frac{eF - fE}{EG - F^2}, \]
\[ a_{22} = \frac{fF - gE}{EG - F^2}. \]

1. Show that at the origin \((0, 0, 0)\) of the hyperboloid \(z = axy\) we have \(K = -a^2\) and \(H = 0\).

2. Consider Enneper’s surface
\[ r(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, v^2 + u^2 - v^2) \]
and show that
(a) the coefficients for the first fundamental form are \(E = G = (1 + u^2 + v^2)^2\) and \(F = 0\),
(b) the coefficients for the second fundamental form are \(e = 2, f = 0,\) and \(g = -2,\) and
(c) the principle curvatures are
\[ k_1 = \frac{2}{(1 + u^2 + v^2)^2} \text{ and } k_2 = -\frac{2}{(1 + u^2 + v^2)^2}. \]

The following theorem is very important in the classification of surfaces.

**Theorem 0.1 (Bonnet)** Let \(E, F, G, e, f, g\) be differentiable functions defined in an open set \(V \subset \mathbb{R}^2\) with \(E > 0\) and \(G > 0\). Assume that the given functions satisfy the Gauss equation,
\[ (\Gamma^2_{12})_u + (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} \]
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and the Mainardi-Codazzi equations,
\[ e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2 \]
and
\[ f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2, \]
and that \( EG - F^2 > 0 \). Then, for all \( q \in V \) there exists a neighborhood \( U \subset V \) of \( q \) and a diffeomorphism \( r : U \to r(U) \subset \mathbb{R}^3 \) such that the regular surface \( r(U) \subset \mathbb{R}^3 \) has \( E,F,G \) and \( e,f,g \) as the coefficients of the first and second fundamental forms, respectively. Furthermore, if \( U \) is connected and if \( \bar{r} : U \to \bar{r}(U) \subset \mathbb{R}^3 \) is another diffeomorphism satisfying the same conditions, then there exist a translation \( T \) and a proper linear orthogonal transformation \( O \) in \( \mathbb{R}^3 \) such that \( \bar{r} = T \circ O \circ r \).

3. Show that no neighborhood of a point in a sphere may be isometrically mapped into a plane.

4. Show that there exists no surface \( r(u,v) \) such that \( E = G = 1 \), \( F = 0 \) and \( e = 1 \), \( g = -1 \), and \( f = 0 \).

5. Does there exist a surface \( r(u,v) \) with \( E = 1 \), \( F = 0 \), \( G = \cos^2 u \) and \( e = \cos^2 u \), \( f = 0 \), and \( g = 1 \)?

6. Justify why the sphere, cylinder, and saddle \( (z = x^2 - y^2) \) are not pairwise locally isometric.