Homework Assignment 4

January 8, 2014

1. The set $C^\infty(M)$ is a ring under pointwise addition and multiplication. (So there exists an additive identity, addition is closed, there exist additive inverses, there is a multiplication identity, and multiplication is closed.) Prove that $\mathcal{X}(M)$ is a module over $C^\infty(M)$. So if $f, g \in C^\infty(M)$ and $Y, Z \in \mathcal{X}(M)$ we have
   
   (a) $f(Y + Z) = fY + fZ$,
   (b) $(f + g)Y = fY + gY$,
   (c) $(fg)Y = f(gY)$, and
   (d) $1Y = Y$.

2. Prove that $[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$ and for all $f, g \in C^\infty(M)$ we have $[fV, gW] = fg[V, W] + (fVg)W - (gWf)V$.

3. Find $[V, W]$ in $\mathbb{R}^3$ for the following:
   
   (a) $V = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}$ and $W = \frac{\partial}{\partial y}$,
   (b) $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $W = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$.

4. For $M$ a manifold and $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$ two orientations we say they are equivalently oriented if $\varphi'_\beta \circ \varphi^{-1}_\alpha$ has positive Jacobian determinant. Show this is an equivalence relation.

5. Let $f : M \to N$ be a diffeomorphism of connected oriented manifolds. Show that if $df_x : T_xM \to T_{f(x)}N$ preserves orientation at one point $x$, then $f$ preserves orientation globally.