1. Integrate $xe^{-x^2(1+y^2)}$ in two ways over $(0, \infty) \times (0, \infty)$ and conclude from this that

$$\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$$ 

2. On $(0,1) \times (0,1)$ let

$$f(x,y) = \begin{cases} 
  x^{-2} & \text{if } y < x < 1 \\
  -y^{-2} & \text{if } x < y < 1.
\end{cases}$$

Show that $\int_0^1 dx \int_0^1 f(x,y) dy = 1$ and $\int_0^1 dy \int_0^1 f(x,y) dx = -1$. Explain why this does not give a counterexample to Fubini’s Theorem.

3. Let $f : E \to [0, \infty]$ where $E \subset \mathbb{R}^n$ is measurable and assume that $f$ is $\mathcal{L}$-measurable. Think of $(x,y) \in \mathbb{R}^{n+1}$ where $y = f(x)$. Let $B = \{(x,y) : 0 \leq y \leq f(x), x \in E\}$ and $A = \{(x,y) : 0 \leq y < f(x), x \in E\}$. Prove that

(a) $\lambda(B) = \lambda(A) = \int_E f(x) dx$ (Note: this says the graph has zero measure) and

(b) $$\int_E f(x) dx = \int_0^\infty \lambda\{x \in E : f(x) > y\} dy.$$