1. Let $f, g \in L^1(\mathbb{T})$. The convolution of $f$ and $g$ is

$$(f * g)(x) = \int_{\mathbb{T}} f(y) g(x-y) dy.$$ 

Prove the following:

(a) $f * g$ is periodic;
(b) $f * g \in L^1(\mathbb{T})$;
(c) $\int_{\mathbb{T}} f * g dx = \int_{\mathbb{T}} f dx \cdot \int_{\mathbb{T}} g dx$;
(d) $f * g = g * f$;
(e) $f * (g * h) = (f * g) * h$; and
(f) $f * e^{inx} = ce^{inx}$.

2. A character of a topological group $G$ (so $G$ is a topological space, a group, and the group action is continuous) is a continuous homomorphism of $G$ into the unit circle. So $\varphi : G \to \mathbb{T}$ is continuous and

$\varphi(x+y) = \varphi(x)\varphi(y)$ for all $x, y \in G$. Prove the following:

(a) for $\xi \in \mathbb{R}^n$ the function $\varphi(x) = e^{ix\cdot\xi}$ is a character for $\mathbb{R}^n$;
(b) prove any character of $\mathbb{R}^n$ is of the form $\varphi(x) = e^{ix\cdot\xi}$ for some $\xi \in \mathbb{R}^n$; and
(c) prove that the characters of $\mathbb{T}$ are orthogonal, so

$$\int_{\mathbb{T}} e^{inx} \overline{e^{imx}} dx = 0 \text{ if } n \neq m.$$