Measurable Functions

Defn: Let \( f: \mathbb{R}^n \to [-\infty, \infty] \) be measurable.
(Then \( f_+ \) and \( f_- \) are measurable). If \( S f_+ d\lambda \) and \( S f_- d\lambda \) are finite, then \( f \) is integrable and
\[
S f d\lambda = S f_+ d\lambda - S f_- d\lambda
\]

The set of integrable functions is
\[
L^1 \text{ or } L^1(\mathbb{R}^n) \text{ or } L^1(\mathbb{R}^n, \mu, \lambda)
\]

Prop: Assume \( f \) and \( g \) are finite on all of \( \mathbb{R}^n \), \( f, g \in L^1 \) and \( a, b \in \mathbb{R} \). Then \( \int (af + bg) d\lambda = a \int f d\lambda + b \int g d\lambda \).

Pf: To prove this we show \( \int af d\lambda = a \int f d\lambda \) and
\[
\int (f + g) d\lambda = \int f d\lambda + \int g d\lambda.
\]
First, suppose $a > 0$. Then $(af)_+ = af_+$ and $(af)_- = af_-$. So $\int_{a \mathcal{f} \mathcal{d} \lambda} = \int_{af_+ \mathcal{d} \lambda} - \int_{af_- \mathcal{d} \lambda} = \int_{f_+ \mathcal{d} \lambda} - \int_{f_- \mathcal{d} \lambda} = a \left[ \int_{f_+ \mathcal{d} \lambda} - \int_{f_- \mathcal{d} \lambda} \right]$

If $a < 0$, then $(af)_+ = -af_-$ and $(af)_- = -af_+$. So $\int_{a \mathcal{f} \mathcal{d} \lambda} = -\int_{af_- \mathcal{d} \lambda} - \int_{af_+ \mathcal{d} \lambda} = -a \int_{f_- \mathcal{d} \lambda} + a \int_{f_+ \mathcal{d} \lambda} = a \int_{f \mathcal{d} \lambda}$.

Last, let $h = f + g$. Then $h_+ - h_- = h = f + g = (f_+ - f_-) + (g_+ - g_-)$ and $h_+ + f_- + g_- = h_- + f_+ + g_+$. So $\int_{h \mathcal{d} \lambda} + \int_{f_- \mathcal{d} \lambda} + \int_{g_- \mathcal{d} \lambda} = \int_{h_- \mathcal{d} \lambda} + \int_{f_+ \mathcal{d} \lambda} + \int_{g_+ \mathcal{d} \lambda}$.
and
\[ S_{h+d\lambda} - S_{h-d\lambda} = S f_{d\lambda} - S f_{d\lambda} + S g_{d\lambda} - S g_{d\lambda} \]

this is equivalent to
\[ S h d\lambda = S f d\lambda + S g d\lambda. \]

So we have the desired result so long as \( h \in L^1 \).

To see this notice \( h_+ \leq f_+ + g_+ \) and
\[ S h d\lambda \leq S f d\lambda + S g d\lambda < \infty. \] A similar argument shows \( h \in L^1 \) and \( h \in L^1 \).

\[ \text{Thm. (Lebesgue's Dominated Convergence Theorem)} \]

Let \( f_1, f_2, \ldots \) be measurable functions on \( \mathbb{R}^n \).
Assume \( g \geq 0 \) and \( g \in L^1 \). If
\[ \lim_{k \to \infty} f_k \text{ exists } \forall x \in \mathbb{R}^n \text{ and } |f_k(x)| \leq g(x) \forall x \in \mathbb{R}^n, \]
then \( \lim_{k \to \infty} f_k \in L^1 \) and
\[ \int (\lim_{k \to \infty} f_k) d\lambda = \lim_{k \to \infty} \int f_k d\lambda. \]
Pf: Let $f = \lim f_k$. Since $|f| \leq g$ we know $f$ and each $f_k$ are integrable (exercise). From Fatou's lemma we have

$$S(g+f) d\lambda \leq \liminf S(g+f_k) d\lambda \quad \text{and}$$

$$S g d\lambda + S f d\lambda \leq S g d\lambda + \liminf S f_k d\lambda.$$

Since $S g d\lambda \in \mathbb{R}$ we have

$$S f d\lambda \leq \liminf S f_k d\lambda.$$

Using $g-f_k$ instead of $g+f_k$ we have

$$S (-f) d\lambda \leq \liminf S (-f_k) d\lambda$$

$$= \liminf - S f_k d\lambda$$

$$= - \limsup S f_k d\lambda \quad \text{or}$$

$$\limsup S f_k d\lambda \leq S f d\lambda.$$

Hence,

$$\limsup S f_k d\lambda \leq S f d\lambda \leq \liminf S f_k d\lambda.$$
**Almost Everywhere**

**Defn:** A property holds **almost everywhere** it holds except on a null set.

**Example:** Let \( C \) be the Middle-'\( \frac{1}{3} \)'s Cantor set.

Let \( f(x) = \begin{cases} 1 & x \in [0, 1] - C \\ 0 & x \in C \end{cases} \)

Then \( f \) is continuous **almost everywhere** (abbreviated a.e.)

**Remark:** For measurable functions \( f \) and \( g \) we say \( f = g \) a.e. if \( \{ x : f(x) \neq g(x) \} \) is a null set.

This defines an equiv. relation denoted \( f \sim g \) if \( f = g \) a.e.
Properties

1. If $f=g$ a.e. are nonnegative, then $\int f d\lambda = \int g d\lambda$

2. If $f=g$ a.e., then $f \in L^1 \iff g \in L^1$ and
   \[ \int f d\lambda = \int g d\lambda. \]

Remark: If $f$ is defined only a.e., then $f$ is measurable iff its extension to $\mathbb{R}^n$ by defining $f$ to be 0 where undefined is measurable.

2) We can then restate convergence theorems for functions defined a.e.

Thm: (LDCT) Assume $f_1, f_2, \ldots$ are measurable defined a.e. on $\mathbb{R}^n$. If $g$ is defined a.e., $g \geq 0$, $g \in L^1$, and

\[
\lim_{k \to \infty} f_k(x) \text{ exists a.e. for } x \in \mathbb{R}^n, \text{ and } \\
|f_k(x)| \leq g(x) \text{ for a.e. } x \in \mathbb{R}^n, \text{ then } \\
\int (\lim_{k \to \infty} f_k) d\lambda = \lim_{k \to \infty} \int f_k d\lambda. 
\]
"Pf." For each \( f_k, f, g \) there is a null set where we could have a problem. Take the countable union of these and on this larger null set we set everything to be zero and we can apply the "traditional" LDCT.

\[
\text{Thm: Let } f_1, f_2, \ldots \text{ be in } L^1 \text{ and } \sum_{k=1}^{\infty} \int |f_k| \, d\lambda < \infty. \\
\text{Then } \sum_{k=1}^{\infty} f_k(x) \text{ exists for a.e. } x \in \mathbb{R}^n \text{ and}
\]

\[
\int \left( \sum_{k=1}^{\infty} f_k \right) \, d\lambda = \sum_{k=1}^{\infty} \int f_k \, d\lambda.
\]

\[
Pf: \text{Let } g = \sum_{k=1}^{\infty} |f_k|. \text{ So } g \in L^1 \lambda \text{ (look back at result on nonnegative functions) and } \\
\int g \, d\lambda = \sum_{k=1}^{\infty} \int |f_k| \, d\lambda < \infty.
\]

So \( g(x) < \infty \) for a.e. \( x \) and \( \sum_{k=1}^{\infty} |f_k(x)| < \infty \) a.e.

Then \( \sum_{k=1}^{\infty} f_k(x) \) conv. abs. for these values. Set
\[ F_j = \sum_{k=1}^{j} f_k. \] Therefore, \( |F_j| \leq g \) \( \forall j \) and LDT says \( \int S(\lim_{j \to \infty} F_j) \, d\lambda = \lim_{j \to \infty} \int S F_j \, d\lambda. \) \( \Box \)

Examples:
1) Let \( f_k \) be a sequence of nonnegative measurable functions s.t. \( S f_k \, d\lambda \leq 1 \) \( \forall k \) and \( \lim_{k \to \infty} f_k = f \) a.e., then \( S f \, d\lambda \leq 1 \) and \( f \) is integrable. Note even if \( S f_k \, d\lambda = 1 \) \( \forall k \) all that can be said is \( S f \, d\lambda \leq 1. \)

2) Let \( \varphi(x) = \begin{cases} 1 \frac{1}{\sqrt{x}} & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases} \)

\[
\int \varphi \, d\lambda = 4
\]

We will see later that since we can compute the Riemann integral this is integrable and
\[
    f(x) = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-j-1k} \varphi \left( x - \frac{k}{j} \right).
\]

Then
\[
    \int f(x) \, dx = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-j-1k} \cdot 4 = 12
\]

Notice: for \( r \in \mathbb{Q} \), \( \lim_{x \to r} f(x) = \infty \) and yet it is integrable.

Also, since \( \int f(x) \, dx < \infty \) we know \( f(x) < \infty \) a.e.

So
\[
    \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-j-1k} \varphi \left( x - \frac{k}{j} \right) \text{ converges a.e.}
\]