We now give the usual defn. of compactness. In the case of $\mathbb{R}^n$ with distance measured with Euclidean metric we will see defn. is equivalent to sequential compactness. However, in some spaces this is not the case.

**Defn.** Let $A \subseteq \mathbb{R}^n$: An open cover for $A$ is a collection of open sets $\{O_i: i \in I\}$ where $A \subseteq \bigcup_{i \in I} O_i$.

A finite subcover is a finite subcollection from the original cover collection that still covers $A$.

**Example:**
1. $A = (0,1)$, $O_i = (0, \frac{1}{i})$, i.e. $\mathbb{N}$
2. $A = (0,1)$, for each $x \in (0,1)$, let $O_x = (\frac{x}{2}, 1)$
3. $A = [0,1]$, for each $x \in (0,1)$, let $O_x = (\frac{x}{2}, 1)$ and $O_0 = (-\frac{1}{16}, \frac{1}{16})$, $O_1 = (1-\frac{1}{16}, 1+\frac{1}{16})$
4. $A = \mathbb{R}$, $O_i = (i-1, i+1)$, $i \in \mathbb{Z}$.

Q: Which of those have finite subcovers?

(3 only)

Defn: A set $K \subseteq \mathbb{R}^n$ is compact if any open cover of $K$ has a finite subcover.

Thm: (Heine-Borel Thm) Let $K \subseteq \mathbb{R}^n$. Then $K$ is compact iff $K$ is closed and bounded.

Cor: So TFAE:

1) $K$ is compact
2) $K$ is sequentially compact
3) $K$ is closed and bounded.
\textbf{Proof:} Let \( K \subseteq \mathbb{R}^n \) be compact and for each \( x \in K \) let 
\[ O_x = B_1(x). \]
The cover \( \{O_x : x \in K\} \) is an open cover of \( K \) so \( K \) has a finite subcover \( \{O_{x_1}, \ldots, O_{x_m}\} \).

Let \( M = \max \{\|x_1\|, \ldots, \|x_m\|\} \). Then 
\[ K \subseteq B_{M+2}(0) \] and \( K \) is bounded.

Let \( \{u_k\}_k \subseteq K \) and assume \( \lim_{k \to \infty} u_k = u \) where \( u \notin K \).

Let \( O_x = B_{\frac{\|x - u\|_2}{2}}(x) \) and \( \{O_x : x \in K\} \) is an open cover of \( K \). Let \( O_{x_1}, \ldots, O_{x_m} \) be a finite subcover.

Fix \( \varepsilon_0 = \min \left\{ \frac{\|x_i - u\|_2}{2} : 1 \leq i \leq m \right\} \). Fix \( k \in \mathbb{N} \) s.t. 
\[ \|u_k - u\|_2 < \varepsilon_0. \] Then
\[ u_k \notin \bigcup_{i=1}^{m} O_{x_i}. \] Hence, \( K \) is closed.
Let $k$ be closed and bold. Assume $\exists$ an open cover $\{O_i\}_{i \in I}$ with no finite subcover. 

Let $M > 0$ s.t. $[M, M] \times \ldots \times [-M, M] \supseteq K$. Divide as we did before. Then there exists $M_1 \in Z^n$ squares containing pts. in $K$ that cannot be covered by finite subcover. We can keep dividing and a sequence $M_1, M_2, \ldots$ such that each of these contain pts. in $K$ that cannot be covered by a finite subcover. Fixing a sequence $u \in M_k \cap K$ we know this is a bold sequence so has a conv. subsequence $u_k$ to $u \in K$. Hence, $u \in M_k \forall k \in N$. We know $\exists O_{i_0}$ s.t. $u \in O_{i_0}$, hence $\exists \varepsilon > 0$ s.t. $B_{\varepsilon}(u) \subseteq O_{i_0}$. Then for $N$ suff. large we know $M_N \subseteq O_{i_0}$ since size of $M_N = \frac{2M}{2^N}$. \[\Box\]
Compact Sets

Lemma: Let $K \subseteq \mathbb{R}^n$ be compact and $\{G_i : i \in I\}$ be an open cover of $K$. Then there exists $\epsilon > 0$ such that $\forall x \in K$ there exists $i \in I$ such that $B_{\epsilon}(x) \subseteq G_i$.

($\epsilon$ is called a Lebesgue number)

Proof: Let $x \in K$. Then there exists $i(x) \in I$ such that $x \in G_{i(x)}$. So there exists $r(x) > 0$ where $B_{r(x)}(x) \subseteq G_{i(x)}$. Then

$$K \subseteq \bigcup_{x \in K} B_{r(x)}(x)$$

and since $K$ is compact, there exists a finite subcover and $x_1, \ldots, x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^n B_{r(x_i)}(x_i).$$

Let $\epsilon = \min_{1 \leq i \leq n} r(i)$.

Now let $x \in K$. Then there exists $1 \leq j \leq N$ such that $x \in B_{r(x_j)}(x_j)$ and $d(x, x_j) < r(x_j) \leq 2r(x_j) - \epsilon$. So by your homework, $B(x, \epsilon) \subseteq B_{\epsilon}(x) \subseteq B_{2r(x_j)}(x_j) \subseteq G_i(x_j)$. 
Cor: Let \( K \) be compact and \( G \) be open where \( K \subset G \).

Then \( \exists \varepsilon > 0 \) s.t. \( \forall x \in K \) we have
\[
B_{\varepsilon}(x) \subseteq G.
\]

Pf: Apply lemma with \( K \) open set \( G \). \( \square \)

Cor: Let \( K \) be compact and \( F \) be closed with \( K \cap F = \emptyset \).

Then \( \exists \varepsilon > 0 \) s.t. \( \forall x \in K, y \in F \), \( d(x, y) > \varepsilon \).

Pf: Apply previous cor. with \( G = F^c \). \( \square \)

Thm: (Bolzano-Weierstrass) Every bounded infinite subset of \( \mathbb{R}^n \) has a limit point.

Pf: Suppose \( A \subset \mathbb{R}^n \) is bounded and has no limit points.

(We will show this implies \( \# A < \infty \).) Since \( A \) has no limit points, it contains all of them vacuously and \( A \) is closed.

So \( A \) is compact since it is bounded.
Let $x \in A$. Then $x$ is not a limit pt. so $\exists \varepsilon(x) > 0$

s.t. $B_{\varepsilon(x)}(x) \cap A = \{x\}$. Then $A \subset \bigcup_{x \in A} B_x$ and

$A$ is compact $\exists$ a finite subcover $x_1, \ldots, x_N \in A$ where

$A \subset \bigcup_{k=1}^N B_{\varepsilon(x_k)}(x_k)$. But each $x \in A$ is only contained in

$B_{\varepsilon(x)}(x)$ so $A = \{x_1, \ldots, x_N\}$. \qed

Cor: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.