Construction of Lebesgue Measure on $\mathbb{R}^n$

Note: 2 methods of constructing Lebesgue Measure

1) Directly: Pros: Concrete
   Cons: Long and not elegant

2) Existence: Pros: Short and elegant
   Cons: No intuition and abstract

We will use the 1st method

For $A \subseteq \mathbb{R}^n$ we want to measure its size.

$\lambda(A)$ = Lebesgue measure of $A$.

Remark: 1) Not all sets will have a measure defined
2) We will start with easy sets and build up to more complex sets.
Step 0: The $\phi$. We define $\lambda(\phi) = 0$.

Step 1: Special Rectangles: $I = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ (similar to what was done in 342).

$\lambda(I) = (b_1 - a_1) \cdots (b_n - a_n)$.

Remarks: 1) If $a_i = b_i$ for $1 \leq i \leq n$ we know $\lambda(I) = 0$.

2) If a rectangle is rotated we can't yet compute measure, but we will see later it gives the same measure.

Step 2: Special Polygons:

A special polygon is a finite union of special rectangles.
First let $P$ be a special rectangle, and

$$P = \bigcup_{i=1}^{N} I_i,$$

where each $I_i$ is a special rectangle and $I_i \cap I_j = \emptyset$ if $i \neq j$.

Then

$$\lambda(P) = \sum_{k=1}^{N} \lambda(I_k).$$

**Idea:**

1) We want $\lambda$ to be additive on all disjoint unions.

2) The measure of boundaries of rectangles is 0 by previous remark, so overlapping on boundaries should not affect measure.

**Facts:**

1) If $P_1 \subseteq P_2$ and $P_1$ and $P_2$ are special rectangles, then

$$\lambda(P_1) \leq \lambda(P_2).$$

(Idea: Let $P = P_2 - P_1$. This will be a special polygon and $\lambda(P) = 0$. So $\lambda(P_1) = \lambda(P_1) + \lambda(P) = \lambda(P_1)$.)
2) If $P_1 \cap P_2 = \emptyset$, then $\lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$. (clear from the definition)

**Step 3: Open Sets** (This is harder)

Let $G \subseteq \mathbb{R}^n$ be open. If $G \neq \emptyset$, then

$$\lambda(G) = \sup \{ \lambda(P) : P \subseteq G, P \text{ a special polygon} \}$$

**Idea:**

\[ \text{etc.} \]

**Facts**

1) $\lambda(G)$ may be $\infty$ so $0 \leq \lambda \leq \infty$

2) $\lambda(G) = 0 \iff G = \emptyset$

3) $G_1 \subseteq G_2$ where $G_1, G_2$ are open implies $\lambda(G_1) \leq \lambda(G_2)$

4) $\lambda(\mathbb{R}^n) = \infty$

5) $\lambda(\bigcup_{k=1}^{\infty} G_k) \leq \sum_{k=1}^{\infty} \lambda(G_k)$
6) If $\left\{ G_i \right\}_{i=1}^{\infty}$ is a collection of open sets where
$G_j \cap G_k = \emptyset$ for $j \neq k$, then
$$\lambda \left( \bigcup_{i=1}^{\infty} G_i \right) = \sum_{i=1}^{\infty} \lambda (G_i)$$

7) If $P$ is a special polygon, then
$$\lambda (P) = \lambda (P^o).$$

**Proof:**

(1) **Trivial**

(2) If $G \neq \emptyset$, then $\exists$ a nontrivial special polygon contained in $G$
(let $x \in G$ and $\varepsilon > 0$ s.t. $B_\varepsilon (x) \subseteq G$ so $\exists \delta < \varepsilon$ s.t. square
of size $\delta$ centered at $x$ is inside $B_\varepsilon (x) \subseteq G$ and $\lambda (G) \geq \lambda (P) > 0$.

(3) Since $G_1 \subseteq G_2$, we know $\left\{ \lambda (P) : P \subseteq G_1 \right\} \subseteq \left\{ \lambda (P) : P \subseteq G_2 \right\}$
So $\lambda (G_2) \geq \lambda (G_1)$ since it is a LUB of a larger set.

(4) We know $[-M, M] \times \cdots \times [-M, M] \subseteq \mathbb{R}^n \forall M > 0$
So $\lambda (\mathbb{R}^n) \geq (2M)^n$. 
(5) We know $\bigcup_{k=1}^{\infty} G_k$ is open. Let $P$ be a special polygon where $P \subseteq \bigcup_{k=1}^{\infty} G_k$. Since $P$ is compact and $\{G_k\}_{k=1}^{\infty}$ is an open cover there is a Lebesgue number $\varepsilon_0$. Hence, $\forall x \in P \exists k$ s.t. $B_{\varepsilon_0}(x) \subseteq G_k$.

We may assume (by possibly subdividing $P$ finer) that each special rectangle $I_j$, where $P = \bigcup_{j=1}^{\infty} I_j$, has diameter less than $2\varepsilon$. Denote the center of $I_j$ by $x_j$. So $I_j \subseteq B_{\varepsilon_0}(x_j) \subseteq G_k$, for some $k$.

So each $I_j$ is completely contained in at least one $G_k$. Let $P_i$ be union of $I_j$'s s.t. $I_j \subseteq G_i$, and $P_k$ for $2 \leq k \leq n$ be union of $I_j$'s s.t. $I_j \subseteq G_k$ and $I_j \not\subseteq G_1, G_2, \ldots, G_{k-1}$. So $P = \bigcup_{k=1}^{n} P_k$ and each $I_j$ is contained one of the $P_k \subseteq G_k$. So

$$\lambda(P) = \sum_{k=1}^{n} \lambda(P_k) \leq \sum_{k=1}^{\infty} \lambda(G_k).$$

Since $P$ is arbitrary

we have $\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k)$. Note at most $\aleph_0$ of $P_k$ are nonempty.
(6) Suppose $P_1, \ldots, P_n$ are arbitrary special polygons s.t. $P_i \subseteq G_i$ for $1 \leq i \leq n$. Then $P = \bigcup_{k=1}^n P_k$ since $P_k$'s are disjoint we have

$$\lambda(P) = \sum_{k=1}^n \lambda(P_k) \leq \bigcup_{k=1}^n \lambda(G_k).$$

Since $P_k \subseteq G_k$ is arbitrary we have

$$\sum_{k=1}^n \lambda(G_k) \leq \lambda\left(\bigcup_{k=1}^n G_k\right).$$

Since $N$ is arbitrary we have

$$\sum_{k=1}^{\infty} \lambda(G_k) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right).$$

The reverse inequality follows from (5).

(7) For $\epsilon > 0$ there exists a generalized rectangle $R$ s.t. $R \subseteq I$ and $\lambda(R) \geq \lambda(I) - \epsilon$. So $\lambda(I) \geq \lambda(R) > \lambda(I) - \epsilon$. Since $\epsilon$ is arbitrary we know $\lambda(I) \geq \lambda(I)$.
Now let $P$ be a special polygon composed of
\[ \left\{ I_k \right\}_{k=1}^{N}, \] nonoverlapping special rectangles
s.t. $P = \bigcup_{k=1}^{N} I_k$, then $P^o \supset \bigcup_{k=1}^{N} I_k^o$ so (6)
implies $\lambda(P^o) \geq \sum_{k=1}^{N} \lambda(I_k^o) \geq \sum_{k=1}^{N} \lambda(I_k) = \lambda(P)$.

Also, if $Q$ is another a special polygon s.t. $Q \subseteq P^o$ we know $Q \subseteq P$ and so $\lambda(Q) \leq \lambda(P)$. Since $Q$ is arbitrary we know $\lambda(P^o) \leq \lambda(P)$.

\[ \square \]

Remark: This is already an improvement from 342. We could find volume of open sets if boundary had Jordan content zero, but there are open sets whose boundary does not have Jordan content zero.

(Interior of Koch's snowflake)