Borel Sets

Defn: The Borel sets in $\mathbb{R}^n$ is the $\sigma$-algebra generated by open sets in $\mathbb{R}^n$ using the Euclidean topology.

Note: If we let $\mathcal{B} =$ Borel sets. Then $\mathcal{B} \subseteq \mathcal{L}$ since $\mathcal{L}$ is a $\sigma$-algebra containing the open sets.

Defn: A set $A \subseteq \mathbb{R}^n$ is a null set if $A \in \mathcal{L}$ and $\lambda(A) = 0$.

Thm: Let $A \in \mathcal{L}$. Then $A$ be be decomposed in the following manner

- $A = E \cup N$
- $E$ is a Borel set
- $N$ is a null set
- $E \cap N = \emptyset$
Pf: For each $k \in \mathbb{N}$ there exists a closed set $F_k \subseteq A$ s.t. $\lambda(A \cap F_k) < \frac{1}{k}$. Let $E = \bigcup_{k=1}^{\infty} F_k$. Then $E \in \mathcal{B}$ and for each $k \in \mathbb{N}$ we know $\lambda(A \cap E) < \frac{1}{k-1}$. So $\lambda(A \cap E) = 0$. Define $N = A \cap E$. □

Thm: Let $E$ be a Borel set and $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be continuous. If $A \subseteq \mathbb{R}^m$ is Borel, then $f^{-1}(A)$ is Borel.

Pf: Since Borel sets are not constructive in general we have to use a direct proof using the defn. Let $M = \{ A \subseteq \mathbb{R}^m : f^{-1}(A) \in \mathcal{B}_n \}$, Borel sets in $\mathbb{R}^n$. We want to show $B_m = \{ \text{Borel sets in } \mathbb{R}^m \} \subseteq M$. To do this we show $M$ is a Borel set containing all open sets so contains $B_m$. 
(a) \( f^{-1}(\emptyset) = \emptyset \in \beta_n \). So \( \emptyset \in M \)

(b) Let \( A_k \in M \) for \( k \in \mathbb{N} \). Then \( f^{-1}(A_k) \in \beta_n \) \( \forall k \)

and \( f^{-1}(\bigcup_{k=1}^{\infty} A_k) = \bigcup_{k=1}^{\infty} f^{-1}(A_k) \in \beta_n \) so \( \bigcup_{k=1}^{\infty} A_k \in M \).

(c) Let \( A \in M \). So \( f^{-1}(A) \in \beta_n \). Then

\[ f^{-1}(A^c) = E \cap f^{-1}(A) \in \beta_n \] and \( A^c \in \beta_n \).

So \( M \) is a \( \sigma \)-algebra.

Since \( f \) is continuous we know if \( A \subset \mathbb{R}^m \) is open, that \( f^{-1}(A) \) is open. So \( E \cap H \in \beta_n \) where \( H \) is open.

So \( E \cap H \in \beta_n \) and \( A \in M \). \( \Box \)

Cor: Let \( E \subset \mathbb{R}^n \), \( F \subset \mathbb{R}^m \) be Borel and \( E \xrightarrow{f} F \) be a homeomorphism. If \( B \subset E \), then \( B \in \beta_n \iff f(B) \in \beta_m \).

Pf: Immediate from previous theorem.
A Meas. Set That is Not Borel

Theorem: $\beta \neq 2$.

Pf: We will show this for $n=1$. The proof can be extended to $\mathbb{R}^n$.

Let $C$ be middle-thirds Cantor set and $f$ be the Lebesgue function. Define $g(x) = x + f(x)$ for $0 \leq x \leq 1$. Then

- $g(0) = 0$
- $g(1) = 2$
- $g$ is strictly increasing

So $g: [0,1] \to [0,2]$ is a homeomorphism.

For an interval $I_r$ in $[0,1]\setminus C$ (where $r = \frac{m}{2^n}$) we have $g(x) = x + r$. So $g$ takes an interval
Jo into another interval of the same length.

\[ \lambda(g(C)) = \lambda\left(\bigcup_{r} g(J_r)\right) \]

\[ = 2 - \sum_{r} \lambda(g(J_r)) \]

\[ = 2 - \sum_{r} \lambda(J_r) = 2 - 1 = 1. \]

Now \( d(c) = 0 \). Since \( g(c) \) has pos. meas. \( \exists \) \( B \subset g(c) \) s.t. \( B \notin Z \). Let \( A = g^{-1}(B) \). So \( A \subset C \) and \( \lambda(A) \leq \lambda(C) = 0 \). So \( A \) is a null set and \( A \Subset Z \). However \( A \notin \beta \). Indeed, if \( A \Subset \beta \) then previous corollary says \( g(A) = B \Subset \beta \).