Measurable Functions

We now want to look at the natural class of functions for Lebesgue measure.

Reminder: 341.342

When integrating the class was bold. fn's. defined over a Jordan domain with discontinuities having Jordan content zero.

We want a much larger class of functions.

Defn: A function $f: X \to [-\infty, \infty]$, where $X \subseteq \mathbb{R}^n$ and $\mathcal{M}$ is a $\sigma$-algebra on $X$, is $\mathcal{M}$-measurable if $\forall t \in [-\infty, \infty]$ the set $f^{-1}([-\infty, t])$ belongs to $\mathcal{M}$.

Before we get to examples we prove the following:
Prop: Let $\mathcal{M}$ be a $\sigma$-algebra of subsets of $\mathbb{X}$ and $f : \mathbb{X} \to [-\infty, \infty]$. Then $f$ is $\mathcal{M}$-measurable iff any one of the following hold:

1. $f^{-1}([-\infty, t]) \in \mathcal{M}$ for all $t \in [-\infty, \infty]$
2. $f^{-1}([t, \infty]) \in \mathcal{M}$ for all $t \in [-\infty, \infty]$
3. $f^{-1}((t, \infty]) \in \mathcal{M}$ for all $t \in [-\infty, \infty]$
4. $f^{-1}((t, \infty)) \in \mathcal{M}$ for all $t \in [-\infty, \infty]$
5. $f^{-1}([-\infty, 3]), f^{-1}([3, \infty)) \in \mathcal{M}$ and $f^{-1}(E) \in \mathcal{M}$ for every Borel set $E \subset \mathbb{R}$.

Pf: $1 \Rightarrow 4, 2 \Rightarrow 3$ are proved by complements which are in $\mathcal{M}$ since it is a $\sigma$-algebra.

Now if $f(x) < t \iff \exists r \in \mathbb{Q}$ s.t. $f(x) \leq r < t$. So

$$f^{-1}([-\infty, t)) = \bigcup_{r \in \mathbb{Q}, r < t} f^{-1}([-\infty, r]) \in \mathcal{M}$$

so $1 \Rightarrow 2$ and

$$f^{-1}([-\infty, t]) = \bigcap_{r \in \mathbb{Q}, r > t} f^{-1}([-\infty, r]) \in \mathcal{M}$$

so $2 \Rightarrow 1$. 

Hence $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$. For $5 \Rightarrow 1$ just let $E = (-\infty, t)$.

Now assume 1-4 hold we will show this implies 5. Notice if $t = -\infty$ in 1 we have $f^{-1}(\{\cdot, \infty\}) \in M$ and $t = \infty$ in 3 gives $f^{-1}(\{\cdot, \infty\}) \in M$.

Let $S = \{E \subseteq \mathbb{R} : f^{-1}(E) \in M\}$. As we did previously we can see $S$ is a $\sigma$-algebra. For $(a, b)$ an open interval $f^{-1}((a, b)) = f^{-1}([-\infty, b)) \cap f^{-1}((a, \infty)) \in M$ since this $\sigma$ is an intersection of 2 sets in $M$ and $f^{-1}((a, b)) \in M$. So $(a, b) \in S$. 

**Examples:**

1) Let $A \subseteq X$ and $x_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \not\in A \end{cases}$.

Then $x_A$ is $M$-meas iff $\chi_A \in M$. 

2) $f(x) = \begin{cases} 1 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$
Properties: Let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be $M$-meas. Then

1. $M$-m.e. $\psi: \mathbb{R} \to \mathbb{R}$ Borel meas. implies $\psi \circ f$ is $M$-meas.
2. $f \geq 0$ $M$-meas implies $\frac{1}{f}$ is $M$-meas.
3. $0 < p < \infty$ implies $1/f^p$ is $M$-meas.
4. $fg$ is $M$-meas.
5. $fg$ is $M$-meas.
6. $f_k: X \to [-\infty, \infty]$ $M$-meas $\forall k$. Implies the following are all $M$-meas.
   
   a) $\sup_k f_k$
   b) $\inf_k f_k$
   c) $\limsup_k f_k$
   d) $\liminf_k f_k$
   e) $\lim f_k$ (if it exists)
Pf:

1. If $E \in \mathcal{B}$ is Borel, then $\varphi^{-1}(E)$ is Borel. By Property 5 in the previous proposition we know
   
   $$(\varphi \circ f)^{-1}(E) = f^{-1}(\varphi^{-1}(E)) \in \mathcal{M}.$$ 

2. Exercise

3. Since $\varphi(t) = 1 + t^p$ is continuous it is Borel meas.
   Now $|f|^p = \varphi \circ f$ and the result follows by (1)

4. $f(x) + g(x) < t \iff f(t) < t - g(x) \iff \exists \ r \in \mathbb{Q}$
   s.t. $f(x) < r < t - g(x)$.
   So
   
   \[ \{ x : f(x) + g(x) < t \} = \bigcup_{r \in \mathbb{Q}} f^{-1}((-\infty, r)) \cap g^{-1}((-\infty, t-r)) \]
   
   and right side is in $\mathcal{M}$ since $\mathcal{M}$ is a $\sigma$-algebra 
   and both $f$ and $g$ are $\mathcal{M}$-measurable.

5. Notice
   
   \[ fg = \frac{1}{4} (f+g)^2 - \frac{1}{4} (f-g)^2. \]
   Now use (4) and (1) to obtain the result.
6. Note \( \{ x : \sup_k f_k(x) \leq t \} = \bigwedge_k \sup_x f_k(x) \leq t \).

So \( \sup f_k \) is \( M \)-meas. Similarly, \( \inf f_k \) is \( M \)-meas.

Also, \( \limsup f_k = \inf \{ \sup f_k \} \). So
\[
\forall j \geq 1 \quad \exists k \geq j
\]
this is \( M \)-meas. Likewise, \( \liminf f_k \) is \( M \)-meas and \( \lim f_k = \limsup f_k = \liminf f_k \) provided these exist and are equal. \( \Box \)
Simple Functions:

Defn: A simple function from $X$ to $[-\infty, \infty]$ is any function that only takes a finite number of values. So $S = \sum_{k=1}^{m} a_k \chi_{A_k}$ where the $A_k$ are disjoint and $a_k \in [-\infty, \infty]$ are distinct.

Note: 1) A simple function is $\mu$-measurable iff each set $A_k \in \mathcal{M}$.

2) If $S_1$ and $S_2$ are simple. Then the following are simple:
   a) $S_1 + S_2$
   b) $S_1 - S_2$
   c) $S_1 \cdot S_2$
   d) $15,1^p$
For $a \in [-\infty, \infty]$ we let
\[ a_+ = \begin{cases} a & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases} \quad \text{and} \quad a_- = \begin{cases} 0 & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \]

So
1) $a = a_+ - a_-$
2) $|a| = a_+ + a_-$
3) $a_+ a_- = 0$

For $f : X \to [-\infty, \infty]$ we let $f_+$ and $f_-$ be defined by $f_+(x) = (f(x))_+$ and $f_-(x) = (f(x))_-$. 

Exercise: If $f$ is $M$-meas., then $f_+$ and $f_-$ are $M$-meas.
Thm: Let \( f: X \to [-\infty, \infty] \) be \( M \)-meas. Then there exists a sequence \( S_1, S_2, \ldots \) of \( M \)-meas. simple functions s.t. \( \lim_{k \to \infty} S_k = f \) on \( X \). Furthermore, if \( f \geq 0 \), then we may choose the simple functions s.t. \( 0 \leq S_1 \leq S_2 \leq \ldots \) and more generally we can have \( 1 S_1 \leq S_2 \leq S_3 \leq \ldots \).

Pf: Suppose \( f \geq 0 \). Define \( S_k(x) = \begin{cases} \frac{i-1}{2^k} & \text{if } \frac{i-1}{2^k} \leq f(x) < \frac{i}{2^k} \\ k & \text{if } k \leq f(x) \end{cases} \) for \( i = 1, \ldots, 2^k k \).

Since \( f \) is \( M \)-meas. we know \( f^{-1} \left( \left[ \frac{i-1}{2^k}, \frac{i}{2^k} \right) \right) \) and \( f^{-1} \left( \{ k \} \right) \) are in \( M \). The result now follows from previous properties of \( M \)-meas. functions.

If \( f \) is a general function choose \( 0 \leq S_1 \leq S_2 \leq \ldots \) converging to \( f_+ \) and \( 0 \leq S_1'' \leq S_2'' \leq \ldots \) converging to \( f_- \) and define \( S_k = S_k' - S_k'' \).
Thm: Suppose $f: \mathbb{R}^n \to [-\infty, \infty]$ is Lebesgue meas.

Then there exists a Borel measurable function $g$ s.t.

$$\{x \in \mathbb{R}^n : f(x) \neq g(x)\}$$

is a null set.

Pf: This follows since any measurable set can be divided into a Borel set and a null set.

Assume $f \geq 0$. Let $0 \leq s_1 \leq s_2 \leq \ldots$ be simple functions s.t.

$$\lim_{k \to \infty} s_k = f.$$ 

We know each $s_k = \sum_{j=1}^{m_k} d_j X_{A_j}$,

where each $A_j \in \mathcal{L}$. Let $A_j = E_j \cup N_j$ where $E_j$ is Borel, $N_j$ is a null set, and $E_j \cap N_j = \emptyset$.

Define $0_k = \sum_{j=1}^{m_k} d_j X_{E_j}$.

So $0 \leq 0_k \leq s_k$ and

$0_k = s_k$ except on a null set $N_k$. Define

$N = \bigcup_{k=1}^{\infty} N_k$ and $g = \sup \sigma_k$. Then $0 \leq g \leq f$ except on $N$. Moreover, by properties proved earlier we know $g$ is Borel measurable.
For \( f \) a general function we pick \( \mathcal{B} \) Borel functions \( 0 \leq g_1 \leq f \) and \( 0 \leq g_2 \leq f \) where \( g_1 = f \) except on a null set and \( g_2 = f \) except on a null set. Let \( g = g_1 - g_2 \). (Note \( \forall x \in \mathbb{R}^n \) either \( g_1(x) = 0 \) or \( g_2(x) = 0 \) so we do not have \( \infty - \infty \).)