Sign the following pledge below:
I pledge on my honor that I have not given or received any unauthorized assistance on this examination.

SIGNATURE: ____________________________

Note that the first 7 questions are true-false. Mark A for true, B for false. These questions are worth 3 points each. Answers to the remaining questions should be written directly on the exam, and should be written neatly and correctly. The points for these remaining questions are indicated on the exam.

True-false questions

1. If $A$ is an $n \times n$ invertible matrix and $F : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $F(x) = Ax$, then $A$ is stable.

2. A function $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable if it is differentiable and is continuous.

3. If $f : X \to Y$ is a continuous map between metric spaces and $K$ is a closed subset of $X$, then $f(K)$ is a closed subset of $Y$.

4. If $F(x, y) = (\cos(x^2+y^2), \sin(x^2+y^2))$, then for every $(x, y) \in \mathbb{R}^2$ we know that $F$ has a local inverse function defined on an open set containing $F(x, y)$.

5. A function $F : \mathbb{R}^m \to \mathbb{R}^m$ is continuous if and only if the preimage of an open set $V$ in $\mathbb{R}^m$ is open in $\mathbb{R}^m$ (so $F^{-1}(V)$ is open in $\mathbb{R}^m$).

6. If $K \subset \mathbb{R}^n$ and $x$ is a boundary point of $K$, then $x$ is a limit point of $K$.

7. If $K \subset \mathbb{R}^n$ is sequentially compact and $F : K \to \mathbb{R}^m$ is continuous, then $F(K)$ is sequentially compact.
8. (11 points) Define the terms in boldface by completing the sentence.

(a) A **metric space** is

   *Look these up.*

(b) A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at a point $x \in \mathbb{R}^n$ if

(c) A set $\mathcal{O} \subset \mathbb{R}^n$ is **open** provided

(d) A set $K \subset \mathbb{R}^n$ is **sequentially compact** if

(e) A set $K \subset \mathbb{R}^n$ is **compact** if
9. (10 points) Consider the following function defined from $\mathbb{R}^2$ to $\mathbb{R}^2$:

\[ f(x, y) = (e^{5x+y}, \sin(\pi x)) \]
\[ g(x, y) = (x^2 y, 3x + 4y) \]
\[ h(x, y) = g(f(x, y)). \]

Compute the derivative matrix of $h$ at the origin using the chain rule.

\[ \mathbf{D}h(0,0) = \mathbf{D}g(f(0,0)) \cdot \mathbf{D}f(0,0) \]

\[ \mathbf{D}f = \begin{bmatrix} 5 e^{5x+y} & e^{5x+y} \\ \pi \cos(\pi x) & 0 \end{bmatrix} \Rightarrow \mathbf{D}f(0,0) = \begin{bmatrix} 5 & 1 \\ \pi & 0 \end{bmatrix} \]

\[ f(0,0) = (e^{5(0)+0}, \sin(\pi 0)) = (1, 0) \]

\[ \mathbf{D}g = \begin{bmatrix} 2xy & x^2 \\ 3 & 4 \end{bmatrix} \Rightarrow \mathbf{D}g(1,0) = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} \]

\[ \mathbf{D}h(0,0) = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ \pi & 0 \end{bmatrix} = \begin{bmatrix} 15 + 4\pi & 3 \\ \pi & 0 \end{bmatrix} \]
10. (10 points) Let \( f(x, y, z) = 3x^2 + x^3 + y^2 + xy^2 + z^3 - 3z \). Show that the gradient of \( f \) at \((0, 0, 1)\) is \(0\). Use the second derivative test to see if you can classify if this is a local maximum or local minimum for \( f \).

\[
\nabla f (x, y, z) = (6x + 3x^2 + y^2, 2y + 3xy, 3z^2 - 3)
\]

\[
\nabla f (0, 0, 1) = (6(0) + 3(0)^2 + 0^2, 2(0) + 3(0)(0), 3(1)^2 - 3) = (0, 0, 0)
\]

\[
\nabla^2 f (x, y, z) = \begin{bmatrix}
6 + 6x & 2y & 0 \\
2y & 2 + 2x & 0 \\
0 & 0 & 6z
\end{bmatrix}
\]

\[
\nabla^2 f (0, 0, 1) = \begin{bmatrix}
6 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 6
\end{bmatrix}
\]

\[
\det \left[ \begin{bmatrix}
6 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 6
\end{bmatrix} \right] = 6 \\
\det \left[ \begin{bmatrix}
6 & 0 \\
0 & 2
\end{bmatrix} \right] = 12 \quad \Rightarrow \quad \nabla^2 f \text{ is pos. definite and}
\]

\[
\det \left[ \begin{bmatrix}
6 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 6
\end{bmatrix} \right] = 72 \\
(0, 0, 1) \text{ is a local minimum.}
\]
11. (12 points) Let \( f: B_r(0) \to \mathbb{R} \) where \( B_r(0) \) is the ball of radius \( r \) centered at the origin and contained in \( \mathbb{R}^n \). Suppose that \( \alpha > 1 \) and \( |f(x)| \leq \|x\|^\alpha \) for all \( x \in B_r(0) \). Prove that the partial derivatives of \( f \) exist at the origin.

Pf: First notice \( f(0) = 0 \). Let \( 1 \leq i \leq n \). Then

\[
\frac{\partial f}{\partial x_i} \left|_{x=0} \right| = \lim_{t \to 0} \frac{f(te_i) - f(0)}{t} = \lim_{t \to 0} \frac{f(te_i)}{t} \text{ provided the limit exists.}
\]

By hypothesis \( |f(te_i)| \leq \|te_i\|^\alpha = 1t^\alpha \|e_i\|^\alpha = t^\alpha \|

where \( \alpha > 1 \). So \( \frac{|f(te_i)|}{1+t} \leq t^\alpha-1 \).

Since

\[
\lim_{t \to 0} -t^\alpha-1 = 0 = \lim_{t \to 0} t^\alpha-1 \quad \text{the squeeze limit theorem}
\]

Says

\[
\lim_{t \to 0} \frac{f(te_i)}{t} = 0 \text{ and } \lim_{t \to 0} \frac{\partial f}{\partial x_i} (0) = 0 \quad \forall 1 \leq i \leq n.
\]
12. (12 points) Let $X$ be a complete metric space and $Y$ be a subset of $X$. Then $Y$ is a complete metric space if and only if $Y$ is a closed subset of $X$.

This is Thm. 12.19 on page 323
13. (12 points) Suppose that \( f : \mathbb{R}^2 \to \mathbb{R} \) and both partial derivatives of \( f \) exist at each point in \( \mathbb{R}^2 \). Suppose also that the partial derivatives are bounded: that is, there exists some \( M > 0 \) such that \( |\frac{\partial f}{\partial x}(p)| \leq M \) and \( |\frac{\partial f}{\partial y}(p)| \leq M \) at every point \( p \) in \( \mathbb{R}^2 \). Prove that \( f \) is continuous.

\[
\text{Pf: Let } p \in \mathbb{R}^2. \text{ Then}
\]

\[
f((x+h) - f(x) = h_1 \frac{\partial f}{\partial x}(z_1) + h_2 \frac{\partial f}{\partial y}(z_2)
\]

where \( \|p - z_i\| \leq \|h\| \) for \( i = 1 \) or \( 2 \) by the Mean Value Proposition. Then

\[
|f(p+h) - f(p)| \leq \|h_1\| M + \|h_2\| M = (\|h_1\| + \|h_2\|) M.
\]

Hence,

\[
\lim_{h \to 0} |f(p+h) - f(p)| \leq \lim_{h \to 0} (\|h_1\| + \|h_2\|) M = 0 \quad \text{and}
\]

\( f \) is cont. at \( p \).
14. (12 points) Prove the following: Let $A$ be a subset of $\mathbb{R}^n$ and suppose that the mapping $F : A \rightarrow \mathbb{R}^m$ is continuous. If the domain $A$ is sequentially compact, then the mapping $F : A \rightarrow \mathbb{R}^m$ is uniformly continuous.

Proof: Let $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ be sequences in $A$ such that $x_k \rightarrow x$ and $y_k \rightarrow y$ for some $x, y \in A$. We need to show that $F(x_k) \rightarrow F(x)$ and $F(y_k) \rightarrow F(y)$.

Look at HWk 7: Problem 11.7: 5