Encode your BYU ID in the grid below.

Instructions

I) Do not write on the barcode area at the top of each page, or near the four circles on each page.

II) Do not talk to other students about this exam until it is over.

III) Fill in the correct boxes for your BYU ID and for the correct answer on the true/false completely.

IV) For questions which require a written answer, show all your work in the space provided and justify your answer.

V) Simplify your answers.

VI) No calculators, books or notes are allowed.

VII) There is no time limit on this exam.
Part I: True/False questions: Mark true if the statement is always true. Otherwise mark false. (4 points each)

1. [ ] True [ ] False Two real numbers $a$ and $b$ satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

2. [ ] True [ ] False Let $B$ be a countable set and let $A \subseteq B$. Then $A$ is finite or countable.

3. [ ] True [ ] False Given any set $A$, there exists a map from $A$ to the power set of $A$ that is onto.

4. [ ] True [ ] False If $a_n \to 0$ and $b_n \to 3$, then $\left(\frac{a_n}{b_n}\right)$ converges.

5. [ ] True [ ] False If there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

6. [ ] True [ ] False Suppose $(a_n) \to a$. There is always an $\epsilon$-neighborhood of $a$ that contains the entire sequence.

7. [ ] True [ ] False If a sequence $(a_n)$ does not converge, then there must be at least two subsequences that converge to different values.

8. [ ] True [ ] False The Cauchy Condensation Test is a way to tell if a particular sequence is a Cauchy sequence.

9. [ ] True [ ] False If a sequence $(a_n)$ is a Cauchy sequence, then every subsequence of $(a_n)$ is also a Cauchy sequence.

10. [ ] True [ ] False There exists a set $B \subseteq \mathbb{R}$ such that $\inf B \geq \sup B$. 


Part II: Give a complete proof of each of the following or show why a proof does not exist.

Prove the Nested Interval Property: For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each $I_n$ contains $I_{n+1}$. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ldots$$

has a nonempty intersection; that is,

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$
If \( A_1, A_2, \ldots, A_m \) are each countable sets, then show the union
\[
\bigcup_{n=1}^{m} A_n
\]
is countable.
Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets of real numbers, each of which is bounded above. Denote $s_k = \sup A_k$. Prove that if $\bigcup_{n=1}^{\infty} A_n$ is bounded above, then
\[
\sup \left( \bigcup_{n=1}^{\infty} A_n \right) = \sup \{s_k\}.
\]
Show that

\[
\lim_{n \to \infty} \frac{1 - n^2}{3n^2 + 2n} = -\frac{1}{3}.
\]
Let \((b_n)\) be a sequence defined by \(b_1 = 1\) and \(b_{n+1} = 4 - \frac{1}{b_n}\).

a) Show that \((b_n)\) is increasing.

b) Show that \((b_n)\) is bounded (and therefore converges by the Monotone convergence Theorem).

c) Find the limit of this sequence. (Hint: \(b_n\) and \(b_{n+1}\) converge to the same number \(s\).)