SYMBOLIC EXTENSIONS FOR PARTIALLY HYPERBOLIC DYNAMICAL SYSTEMS WITH 2-DIMENSIONAL CENTER BUNDLE

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Abstract. We relate the symbolic extension entropy of a partially hyperbolic dynamical system to the entropy appearing at small scales in local center manifolds. In particular, we prove the existence of symbolic extensions for $C^2$ partially hyperbolic diffeomorphisms with a 2-dimensional center bundle.

1. Introduction

The topological entropy of a system $(X,T)$, denoted $h_{\text{top}}(T)$, is a number that measures the topological complexity of the system. If $\mu$ is an invariant Borel probability measure for a map $T$, then the measure theoretic entropy, denoted $h_{\mu}(T)$, is a number that measures the complexity of the system as seen by the measure $\mu$. See for instance [21, p. 169] for precise definitions.

The variational principle says that if $T$ is a continuous map of a compact metrizable space, then

$$h_{\text{top}}(T) = \sup_{\mu \in \mathcal{M}(T,X)} h_{\mu}(T)$$

where $\mathcal{M}(T,X)$ is the set of invariant Borel probability measures. The entropy function is the map $h : \mathcal{M}(X,T) \to \mathbb{R}$ defined by $h(\mu) = h_{\mu}(T)$.

The entropy function is naturally defined as a limit of a nondecreasing sequence of functions, which estimates the complexity of the system at scales decreasing to zero. For certain such sequences, called entropy structures that are defined below, the manner in which this convergence differs from uniform convergence demonstrates how entropy emerges within the system at finer and finer resolution. We will see that entropy structures are related to the entropy of symbolic extensions of the system.

A subshift, $(Y,S)$, is a closed, shift invariant subset of a full shift over a finite alphabet, and a continuous map $\pi : Y \to X$ is a symbolic extension of $(X,T)$ if $\pi \circ S = T \circ \pi$ (we also call $(X,T)$ the factor of the subshift $(Y,S)$). We note that the shift $(Y,S)$ need not be a shift of finite type and the map $\pi$ need not be finite-to-one. Recall also that a factor of a subshift has finite topological entropy.

A natural question regarding symbolic extensions is the following:

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**Question 1.1.** When does a system \((X, T)\) with finite topological entropy have a symbolic extension? Furthermore, what can be said about the symbolic extensions when one does exist?

Using results of Boyle, Fiebig, and Fiebig in [6] and Buzzi in [11] one can show that any \(C^\infty\) diffeomorphism of a compact smooth manifold to itself has a symbolic extension given by a factor map which preserves entropy for every invariant measure, such an extension is called a *principal symbolic extension*.

Another class of dynamical systems for which very nice symbolic extensions are known to exist are Anosov diffeomorphisms. Indeed Markov partitions allow finite to one symbolic extensions which are also subshifts of finite type. As a reminder if we let \(M\) be a compact smooth manifold and \(f : M \to M\) be a diffeomorphism, then \(f\) is *Anosov* if the tangent bundle \(TM\) splits into invariant uniformly contracting and uniformly expanding directions.

In this paper we address the existence of symbolic extensions if we weaken the hyperbolicity and the regularity. Let \(M\) be a smooth compact manifold and \(f : M \to M\) be a diffeomorphism. We denote the set of \(C^r\) diffeomorphisms from \(M\) to \(M\) by \(\text{Diff}^r(M)\) and we endow this space with the usual topology. We say \(f \in \text{Diff}^r(M)\) is *partially hyperbolic* if the tangent bundle \(TM\) has a \(Df\)-invariant splitting \(E^s \oplus E^c \oplus E^u\) and constants 

\[ \lambda_0, \lambda_1, \mu_0, \mu_1 \]

with 

\[ 0 < \lambda_0 < \lambda_1 < 1 < \mu_1 < \mu_0 \]

such that

- \(E^s\) is uniformly contracting: \(\|Df_p(v)\| < \lambda_0 \|v\|\) for all \(p \in M\) and \(v \in E^s\),
- \(E^u\) is uniformly expanding: \(\mu_0 \|v\| < \|Df_p v\|\) for all \(p \in M\) and \(v \in E^u\), and
- \(E^u\) and \(E^s\) dominate \(E^c\): \(\lambda_1 \|v\| < \|Df_p v\| < \mu_1 \|v\|\) for all \(p \in M\) and \(v \in E^c\).

**Main Theorem.** *Let \(f \in \text{Diff}^2(M)\) be partially hyperbolic with a splitting \(E^s \oplus E^c \oplus E^u\). If \(\dim(E^c) = 2\), \(\mu_0^{-1} \mu_1^2 < 1\), and \(\lambda_0 \lambda_1^2 < 1\) where the constants are defined as above, then \(f\) has a symbolic extension.*

The above theorem will be an immediate consequence of Theorem 8.1 in which estimates will be given to not only show the existence of symbolic extensions, but also give an upper bound on the least topological entropy of symbolic extensions.

When the center bundle is one dimensional or splits in a dominated way into one dimensional center bundles then there exist principal symbolic extensions (such systems are also \(h\)-expansive, see Subsection 2.2 below) [12, 25, 13, 14].

Following T.Downarowicz and S.Newhouse [18] L. J. Díaz and the second author of the present paper have shown in [13] that among certain \(C^1\) open sets, \(U\), of partially hyperbolic diffeomorphisms with a 2-dimensional center bundle there is
a $C^1$ residual set, $\mathcal{R}$, contained in $\mathcal{U}$ such that each diffeomorphism in $\mathcal{R}$ has no symbolic extension. This shows there is a stark difference for symbolic extensions between the $C^1$ and $C^2$ settings. In fact, Downarowicz and Newhouse have made the following conjecture:

**Conjecture 1.2.** [18] Every $C^r$ diffeomorphism of a compact manifold to itself has a symbolic extension for $2 \leq r \leq \infty$.

The arguments of this paper are similar to those used by the first author in [7] where it is shown that any $C^2$ diffeomorphism of a compact surface has a symbolic extension. The outline of the paper is as follows. In Section 2 we recall the main tools in the entropy theory of symbolic extensions. In Section 3 certain entropy structures are introduced that will be helpful in estimating the symbolic extension entropy function for the partially hyperbolic diffeomorphisms. In Section 4 we review some facts about Lyapunov exponents for partially hyperbolic diffeomorphisms. In Section 5 we discuss foliations for partially hyperbolic diffeomorphisms. In Sections 6 we recall the main technical tool from [7]. In Section 7 we show a local Ruelle inequality for general partially hyperbolic dynamical and in Section 8 we give a proof of the main theorem.

### 2. Entropy structures

In this section we state some relevant facts on entropy. We note that throughout the rest of the paper we assume that $X$ is a compact metric space and $T : X \to X$ is continuous.

#### 2.1. Bowen balls, separated and spanning sets.

Let $(X, d)$ be a compact metric space and $T$ be a continuous self-map of $X$. For all $n \in \mathbb{N}$, $\epsilon > 0$ and $x \in X$, we define the $\epsilon$-forward Bowen ball $B^+_T(x, n, \epsilon)$ of length $n$ at $x$ as

$$B^+_T(x, n, \epsilon) := \{ y \in X : d(T^kx, T^ky) \leq \epsilon \text{ for all } 0 \leq k < n \}.$$  

Such sets are also called finite forward Bowen balls. We also consider the infinite $\epsilon$-forward Bowen ball $B^+_T(x, \infty, \epsilon)$ at $x$ defined by

$$B^+_T(x, \infty, \epsilon) := \bigcap_{n \geq 1} B^+_T(x, n, \epsilon) = \{ y \in X : d(T^kx, T^ky) \leq \epsilon \text{ for all } k \in \mathbb{N} \}.$$ 

When $T$ is a homeomorphism the intersections

$$B^*_T(x, n, \epsilon) := B^+_T(x, n, \epsilon) \cap B^+_T(x, n+1, \epsilon)$$

and

$$B^*_T(x, \infty, \epsilon) := B^+_T(x, +\infty, \epsilon) \cap B^+_T(x, \infty, \epsilon)$$
are respectively called the $\epsilon$-Bowen ball of length $n$ and the infinite $\epsilon$-Bowen ball at $x$. To simplify the notations we will omit the index $T$ in the above equation when there is no confusion on the map.

For a set $Y \subset X$, a set $A \subset Y$ is $(n, \epsilon)$-separated if $y \notin B^+(x, n, \epsilon)$ for all $x \neq y \in A$. The maximum cardinality of an $(n, \epsilon)$-separated set for $Y$ is denoted $r_n(Y, \epsilon)$ and we let

$$\bar{r}(Y, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r_n(Y, \epsilon).$$

We will also examine spanning sets. A set $A$ is a $(n, \epsilon)$-spanning set of $Y$ if for any $y \in Y$ there exists a point $x \in A$ such that $y \in B^+(x, n, \epsilon)$. There are natural relations between the minimum cardinality of spanning sets and the maximum cardinality of separated sets, see for instance [28].

2.2. $h$-expansivity and tail entropy. Bowen [3] introduced the idea of examining the entropy of points that stay $\epsilon$-close under all forward iterates (a similar notion was first introduced by Misiurewicz for open covers [22]). To be more precise, given $\epsilon > 0$, set

$$h_\star^T(\epsilon) := \sup_{x \in X} \lim_{\delta \to 0} \bar{r}(B^+(x, \infty, \epsilon), \delta).$$

When $T$ is a homeomorphism $h_\star^T(\epsilon) = \sup_{x \in X} \lim_{\delta \to 0} \bar{r}(B^+(x, \infty, \epsilon), \delta)$, see [3]. A map $T$ is $h$-expansive if $h_\star^T(\epsilon) = 0$ for some $\epsilon$. The tail entropy, $h^*(T)$, of a topological dynamical system $(X, T)$ is given by

$$h^*(T) = \lim_{\epsilon \to 0} h_\star^T(\epsilon).$$

This reflects the limit of entropy arising locally on finer and finer scales.

2.3. Entropy structures. With the weak* topology the set $\mathcal{M}(X, T)$ is a compact, convex metric space. The entropy function is usually defined as the limit of a sequence of nondecreasing nonnegative functions. Entropy structures are special examples of these sequences which reflect the convergence of entropy at smaller and smaller scales. Entropy structures define an equivalence class for the following relation:

$$(h_k)_k \equiv (g_k)_k, \text{ when } \forall k \in \mathbb{N}, \forall \gamma > 0, \exists l \in \mathbb{N} \text{ such that } h_l > g_k - \gamma \text{ and } g_l > h_k - \gamma.$$

We refer to [16] for a complete definition. In Subsection 2.5 we explain the link between the convergence of entropy structures and the entropy of the symbolic extensions. Entropy structures are also related with the tail entropy. Indeed, T. Downarowicz [16] proved that the tail entropy satisfies a variational principle.
which claims that for all entropy structures \( H = (h_k)_k \) the tail entropy is the defect of uniform convergence of entropy structures (see also [8]), that is

\[
    h^*(T) = \lim_k \sup \mu (h - h_k)(\mu).
\]

2.4. **Symbolic extension entropy.** To estimate the entropy of symbolic extensions we consider the following quantities. Let \((X, T)\) be a dynamical system. The *extension entropy function* associated with a symbolic extension \( \pi : (Y, S) \to (X, T) \) is the map \( h^*_\text{ext} : \mathcal{M}(X, T) \to \mathbb{R} \) defined by

\[
    h^*_\text{ext}(\mu) = \sup \{ h_\nu(S) : \pi(\nu) = \mu \}.
\]

Then the *symbolic extension entropy function*, \( h_{\text{sex}} \), is defined by

\[
    h_{\text{sex}} = \inf_{\pi} h^*_\text{ext}.
\]

The *topological symbolic extension entropy* of a system \((X, T)\) is

\[
    h_{\text{sex}}(X, T) = \inf \{ h_{\text{top}}(Y, S) : (Y, S) \text{ is a symbolic extension of } (X, T) \}.
\]

The difference \( h_{\text{sex}}(X, T) - h_{\text{top}}(X, T) \) is called the *residual entropy function* and represents the complexity in the system that is "hidden" in the multiscale structure of the system [4].

Finally we recall that the topological symbolic extension entropy and the symbolic extension entropy function are related by the following variational principle [4]

\[
    h_{\text{sex}}(X, T) = \sup_{\mu \in \mathcal{M}(X, T)} h_{\text{sex}}(\mu).
\]

2.5. **Symbolic Extensions Theorem.** We consider a dynamical system \((X, T)\). Let us denote the set of nonnegative upper semicontinuous functions on \( \mathcal{M}(X, T) \) with the infinity function (the constant function equal to \(+\infty\)) by \( \mathcal{S}(X, T) \). Moreover, when \( f \) is a real valued function on \( \mathcal{M}(X, T) \) we denote by \( \lceil f \rceil \in \mathcal{S}(X, T) \) the smallest upper semicontinuous function larger than \( f \) (by convention \( \lceil f \rceil = +\infty \) when \( f \) is unbounded from above). We recall the symbolic extension theorem which connects symbolic extensions and entropy structures. We follow the presentation of [7].

We consider the monotone operator \( T_{\text{sex}} : \mathcal{S}(X, T) \to \mathcal{S}(X, T) \) defined by

\[
    T_{\text{sex}}(f) = \lim_k \lceil f + h - h_k \rceil
\]

where \( (h_k)_k \) is an entropy structure (it is not hard to show that \( T_{\text{sex}} \) does not depend on the choice of the entropy structure). This operator has a smallest fixed point which can be seen as the limit of the transfinite sequence, \( (T_{\text{sex}}^\alpha 0)_{\alpha \uparrow} \), with \( T_{\text{sex}}^\alpha f = T_{\text{sex}}^{\alpha - 1}(T_{\text{sex}} f) \) for a successor ordinal \( \alpha \) and \( T_{\text{sex}}^\beta f = \sup_{\gamma < \beta} T_{\text{sex}}^\gamma f \) for a limit ordinal \( \beta \).
Theorem 2.1. [4][7] (Sex Theorem) The function \( h_{sex} \) is the smallest fixed point of \( T_{sex} \). Moreover the functions \( h_{ext}^\pi \) are exactly the affine fixed points of \( T_{sex} \).

To show that a system \((X, T)\) has a symbolic extension it is therefore enough to find an affine fixed point of \( T_{sex} \). In the following we show that the positive central Lyapunov exponents satisfy this property.

3. Adapted entropy structures

In this section we define new entropy structures which are well adapted to study the convergence of entropy for partially hyperbolic dynamical systems.

We first investigate the behavior of entropy structures with respect to various basic constructions. Namely, we relate the entropy structures of a dynamical system with those of its inverse and powers.

3.1. Entropy structure of the inverse and powers. Entropy structures are invariant by inverting the dynamical system.

Lemma 3.1. Let \((X, T)\) be an invertible dynamical system \((T : X \rightarrow X \text{ is a homeomorphism})\). Then \( \mathcal{H} \) is an entropy structure for \( T \) if and only if \( \mathcal{H} \) is an entropy structure for \( T^{-1} \).

Proof: It is enough to prove the lemma for a particular entropy structure \( \mathcal{H} \). Let \( \mu \in \mathcal{M}(X, T) = \mathcal{M}(X, T^{-1}) \). It follows easily from the invariance of \( \mu \) that

\[
h_T(\mu, P) = h_{T^{-1}}(\mu, P)
\]

for any finite Borel partition \( P \).

As defined in [16] the entropy sequence \( H_{Leb}^{\theta}(T) \) at some invariant measure \( \mu \) is the entropy of \( \mu \times \lambda \) (where \( \lambda \) is the Lebesgue measure of the circle) with respect to a decreasing sequence of partitions with small boundaries (the boundaries have zero \( \mu \)-measure for any invariant measure \( \mu \)) for the product of \((X, T)\) with a given irrational rotation of angle \( \theta \) of the circle. T. Downarowicz proved that \( H_{Leb}^{\theta} \) was an entropy structure for all irrational \( \theta \). It follows from our first observation that

\[
H_{Leb}^{\theta}(T^{-1}) = H_{Leb}^{-\theta}(T).
\]

In particular these two sequences, which are respectively entropy structures for \( T^{-1} \) and \( T \), are uniformly equivalent. \( \square \)

Entropy structures of a product were already studied in [7]. We recall Lemma 1 of [7].

Lemma 3.2. Let \((X, T)\) be a dynamical system with finite topological entropy and let \( \mathcal{H} = (h_k)_{k \in \mathbb{N}} \) be an entropy structure of \( T^p \) with \( p \in \mathbb{N} \). Then the sequence

\[
\frac{1}{p^\alpha} \mathcal{H}_{|\mathcal{M}(X,T)}(\frac{h_k|_{\mathcal{M}(X,T)}}{p})_{k \in \mathbb{N}}
\]

defines an entropy structure of \( T \).

By Lemma 3.1 the above statement also holds true for negative integers \( p \) with \( \frac{1}{|p|} \) instead of \( \frac{1}{p} \).
3.2. Modified Newhouse local entropy. We now give a modified version of Newhouse local entropy. We replace, in the definition due to S. Newhouse [23], the finite forward Bowen ball by the infinite Bowen ball. For partially hyperbolic dynamical systems we will see that this last set is contained in local center manifolds. Therefore in order to estimate the local entropy of such systems we will only need to consider their restriction to local center manifolds.

We recall now the Newhouse local entropy. Let \( x \in X, \epsilon > 0, \delta > 0, n \in \mathbb{N}, \) and \( F \subset X \) a Borel set, we define

\[
H(n, \delta | F, \epsilon) := \sup_{x \in F} r_n \left( F \cap B^+(x, n, \epsilon), \delta \right),
\]

\[
h(\delta | F, \epsilon) := \limsup_{n \to +\infty} \frac{1}{n} H(n, \delta | F, \epsilon), \quad \text{and}
\]

\[
h(X | F, \epsilon) := \lim_{\delta \to 0} h(\delta | F, \epsilon).
\]

Then for any ergodic measure \( \nu \) we define the Newhouse local entropy as

\[
h^{\text{New}}(X | \nu, \epsilon) := \lim_{\sigma \to 1} \inf_{\nu(F) > \sigma} h(X | F, \epsilon).
\]

Finally, we define the harmonic extension and use this to extend the function \( h^{\text{New}}(X| \cdot, \epsilon) \) to \( \mathcal{M}(X, T) \). We now review what is meant by a harmonic extension, see for instance [17] for a more complete description and facts on harmonic extensions. If \( \mu \in \mathcal{M}(X, T) \), then there exists a unique Borel probability measure \( \mathcal{M}_\mu \) on \( \mathcal{M}(X, T) \) supported on the ergodic measures of \( \mathcal{M}(X, T) \), denoted \( \mathcal{M}_\epsilon(X, T) \), such that for all Borel subsets \( B \) of \( X \) we have \( \mu(B) = \int B \, d\mathcal{M}_\mu(\nu) \). This is called the ergodic decomposition of \( \mu \), see for instance [21, p. 139]. A bounded Borel map \( f : \mathcal{M}(X, T) \to \mathbb{R} \) is harmonic if

\[
f(\mu) = \int_{\mathcal{M}_\epsilon(X, T)} f(\nu) \, d\mathcal{M}_\mu(\nu)
\]

for all \( \mu \in \mathcal{M}(X, T) \). All affine upper semicontinuous maps are harmonic. The entropy function is not upper semicontinuous in general, but it is always harmonic. If \( f \) is a bounded real Borel map defined on \( \mathcal{M}_\epsilon(X, T) \), then the harmonic extension of \( f \) is the function \( \tilde{f} \) defined on \( \mathcal{M}(X, T) \) by

\[
\tilde{f}(\mu) := \int_{\mathcal{M}_\epsilon(X, T)} f(\nu) \, d\mathcal{M}_\mu(\nu).
\]

The function \( \tilde{f} \) coincides with \( f \) on \( \mathcal{M}_\epsilon(X, T) \) and is harmonic.

Given a nonincreasing sequence \( (\epsilon_k)_{k \in \mathbb{N}} \) converging to 0, we consider the sequence \( \mathcal{H}_T^{\text{New}} = (h_k^{\text{New}})_{k \in \mathbb{N}} \) with

\[
h_k^{\text{New}} := h - h^{\text{New}}(X| \cdot, \epsilon_k)
\]
for all integers $k$. T. Downarowicz proved this sequence defines an entropy structure [16]. In particular $h^{New}(X|·,\epsilon_k)$ converges pointwise to zero when $k$ goes to infinity.

We will use a modified version of Newhouse local entropy for homeomorphisms that will give us an entropy structure that we can use to help estimate the entropy of symbolic extensions of partially hyperbolic dynamical systems. Essentially, we replace a general Borel set $F$ by compact sets (which makes no difference by the regularity of $\nu$) and the finite forward Bowen ball $B^+(x,n,\epsilon)$ by the infinite two-sided Bowen ball, $B^*(x,\infty,\epsilon)$. We also invert the supremum in $x$ with the logarithmic lim sup in $n$. Intermediate notations will be marked with a star. More precisely, we define for all $x \in X$, $\epsilon > 0$, $\delta > 0$, $n \in \mathbb{N}$ and all compact subsets $F$ of $X$ the following:

$$h^*(\delta|F,\epsilon) := \sup_{x \in F} \bar{r}(F \cap B^*(x,\infty,\epsilon),\delta), \text{ and}$$

$$h^*(X|F,\epsilon) := \lim_{\delta \to 0} h^*(\delta|F,\epsilon).$$

Then for any ergodic measure $\nu$ we put:

$$h^{New}(X|\nu,\epsilon) := \lim_{\sigma \to 1} \inf_{\nu(F) > \sigma} h^*(X|F,\epsilon)$$

and we extend the definition to any invariant measure by the harmonic extension. We see by the definition that $h^{New}(X|\cdot,\epsilon) \leq h^{New}(X|\cdot,\epsilon)$.

Define the sequence $h^{New}_k := h - h^{New}(X|\cdot,\epsilon_k)$. We prove now that $\mathcal{H}^{New}_T = (h^{New}_k)_{k \in \mathbb{N}}$ defines an entropy structure. We follow essentially the proof of Proposition 2.2 of [3].

**Proposition 3.3.** Let $T : X \to X$ be a Lipschitz continuous homeomorphism of a compact metric space $(X,d)$. Then $h^{New}(X|\cdot,\epsilon) = h^{New}_T(X|\cdot,\epsilon)$ for all $\epsilon > 0$. In particular $\mathcal{H}^{New}$ is an entropy structure.

**Proof:** As stated above we know from the definitions that $h^{New}(X|\cdot,\epsilon) \leq h^{New}_T(X|\cdot,\epsilon)$. We now prove the reverse inequality. Let $\sigma \in ]0,1[$ and let $F$ be a compact subset of $X$ of measure larger than $\sigma$. For all ergodic measures $\nu$ we are going to show there exists a Borel set $G$ with measure larger than $\sigma$ such that

$$h(X|G,\epsilon) \leq h^*(X|F,\epsilon) + (1 - \sigma)\text{Lip}(T)$$

where $\text{Lip}(T)$ is a Lipschitz constant for $T$. Let $\alpha > 0$. Fix $x \in F$ and let $n(x)$ be an integer and $E(x)$ be an $(n(x),\delta)$-separated set in $B^+(x,\infty,\epsilon)$ of maximal cardinality such that

$$\frac{1}{n(x)} \log \# E(x) \leq h^*(\delta|F,\epsilon) + \alpha.$$
Then there exist an integer $N(x)$ and $\eta > 0$ such that the finite (two-sided) Bowen ball $B^*(x, N(x), \epsilon + \eta)$ is $(2\delta, n(x))$-spanned by $E(x)$. By compactness of $F$ there exists $x_1, \ldots, x_M \in F$ such that $F$ is covered by the finite two-sided Bowen balls $B^*(x_i, N(x_i), \eta)$. We consider a Borel subset $G$ of $F$ of $\nu$-measure larger than $\sigma$ such that $\frac{1}{n^2}\{0 \leq k < n, T^k x \in F\}$ converges to $\nu(F)$ uniformly in $x \in G$. We fix an integer $N \geq \max(N(x_i), n(x_i))$ such that $\frac{1}{n^2}\{0 \leq k < n, T^k x \in F\} \geq \sigma$ for all $x \in G$ and for all $n \geq N$. Let $n \geq N$ and let $x \in G$. When $T^k x \in F$ with $N \leq k \leq n - N$ there exists $x_i$ such that $T^k x \in B^*(x_i, N(x_i), \eta)$ and then $T^k B(x, n) \in (2\delta, n(x_i))$-spanned by $E(x_i)$. If $T^m x \notin G$ for $m = k, \ldots, k + l$ then $B(y, 2\delta) \cap T^k B(x, n, \epsilon)$ is $(2\delta, l)$-spanned by $\text{Lip}(T)^l$ points for all $y$. It follows that $G \cap B(x, n, \epsilon)$ is $(4\delta, n)$-spanned by

$$\text{Lip}(T)^{(1-\sigma)n+2N}e^{(h^*(\delta F, \epsilon)+\alpha)n}$$

points. Since it holds for all $\delta > 0$ and for all $\alpha > 0$ we get

$$h(X|G, \epsilon) \leq h^*(X|F, \epsilon) + (1 - \sigma)\text{Lip}(T).$$

We easily conclude by taking the infimum over all compact subsets $F$ of $\nu$ measure larger than $\sigma$ and then taking the limit as $\sigma$ goes to 1. \qed

**Remark 3.4.** If $T$ is almost entropy $h$-expansive, i.e. there exists $\epsilon > 0$ such that

$$\lim_{\delta \to 0} \frac{1}{\delta} \log \mu(B^*(x, \infty, \epsilon, \delta)) = 0$$

for $\mu$-almost all $x$ and for all invariant measure $\mu$, then $h_{\text{New}}^*(X|1, \epsilon)$ is zero. Then $T$ is in fact $h$-expansive by the tail variational principle (1). In this way the above proposition implies Proposition 2.5 of [19].

### 4. Positive Lyapunov Exponents of an Invariant Continuous Subbundle

We will see that the bounds on the modified Newhouse local entropy that give the existence of a symbolic extension are related to the Lyapunov exponents of an ergodic measure. More specifically, the bounds will be related to the Lyapunov exponents of the ergodic measure in the center direction. See [26] for the definitions of Lyapunov exponents and related results.

Throughout this section let $M$ be a compact manifold and $T: M \to M$ be $C^1$. We consider a continuous invariant subbundle, $E$, of the tangent bundle of dimension $d$. Let us denote by $\chi^E_\sigma(x) \geq \ldots \geq \chi^E_d(x)$ the Lyapunov spectrum associated to $DT|_E$ of a regular point $x \in M$. Given an invariant measure $\mu$ we consider the sum $\sum_{k} \chi^E_\sigma(\mu)$ of its $k$ first positive Lyapunov exponents in $E$, that is

$$\sum_{k} \chi^E_\sigma(\mu) = \int_M \sum_{i=1}^k \max(\chi^E_i(x), 0)d\mu(x).$$
We prove in this section that this sum defines an upper semicontinuous function on probability invariant measures. We note that it is a well known fact that the positive part of the sum of the $k$ first Lyapunov exponents,

$$\mu \mapsto \max\left(\int_{M} \sum_{i=1}^{k} \chi_{E}^{i}(x) d\mu(x), 0\right)$$

is upper semicontinuous (see for instance [2]). In general these two quantities differ (consider for example a non trivial convex combination of two ergodic invariant measures, one with negative center exponents and the other with positive ones). Therefore, although the arguments are the same we give a proof for completeness.

When $T$ is a $C^1$ map on a manifold $M$ we denote by $\Lambda^{k}D_xT$ the map induced by the differential map $D_xT$ at $x \in M$ on the $k^{th}$ exterior power of the tangent space at $x$.

**Lemma 4.1.** Let $T : M \to M$ be a $C^1$ map on a compact manifold such that the tangent bundle admits a continuous invariant subbundle $E$. Then for any $0 \leq k \leq d$ the sum $\sum_{i=1}^{k} \chi_{E}^{i}$ of the $k^{th}$ largest positive Lyapunov exponents of $T$ in $E$ satisfies

$$\forall \mu \in \mathcal{M}(M, T), \quad \sum_{i=1}^{k} \chi_{E}^{i}(\mu) = \inf_{n \in \mathbb{N}} \frac{1}{n} \int \max_{i=1,\ldots,k} \log^{+} \|\Lambda^{i}D_xT^n|_{E}\| d\mu(x).$$

In particular, $\sum_{i=1}^{k} \chi_{E}^{i} : \mathcal{M}(M, T) \to \mathbb{R}^+$ is upper semicontinuous.

**Proof:** For all integers $n > 0$ we consider the function $g_n : \mathcal{M}(M, T) \to \mathbb{R}^+$ defined by

$$\forall \mu \in \mathcal{M}(M, T), \quad g_n(\mu) = \int \max_{i=1,\ldots,k} \log^{+} \|\Lambda^{i}D_xT^n|_{E}\| d\mu(x).$$

This function is clearly continuous and affine, and therefore harmonic. Also $(g_n(\mu))_{n \in \mathbb{N}}$ is a subadditive sequence for all $\mu \in \mathcal{M}(M, T)$.

According to Osseledet’s Theorem [24][26] we have

$$\sum_{i=1}^{k} \chi_{E}^{i}(\nu) = \lim_{n \to +\infty} \frac{g_n(\nu)}{n}$$

for all ergodic measures $\nu$. Consider now a general measure $\mu \in \mathcal{M}(M, T)$. We have

$$\sum_{i=1}^{k} \chi_{E}^{i}(\mu) := \int_{\mathcal{M}_{\nu}(M, T)} \sum_{i=1}^{k} \chi_{E}^{i}(\nu) dM_{\mu}(\nu)$$

$$= \int_{\mathcal{M}_{\nu}(M, T)} \lim_{n \to +\infty} \frac{g_n(\nu)}{n} dM_{\mu}(\nu).$$
Obviously $g_n(\nu) \leq \log^+ \|DT\|$ for all ergodic measures $\nu$. Therefore by applying the theorem of dominated convergence we get

$$\sum_{k} \chi^E_+(\mu) = \lim_{n \to +\infty} \int_{\mathcal{M}(M,T)} \frac{g_n(\nu)}{n} d\mu(\nu)$$

and by harmonicity of $g_n$ we have

$$\sum_{k} \chi^E_+(\mu) = \lim_{n \to +\infty} \frac{g_n(\mu)}{n}.$$ 

Since the sequence $(g_n(\mu))_{n \in \mathbb{N}}$ is subadditive we also know that

$$\sum_{k} \chi^E_+(\mu) = \inf_{n \in \mathbb{N}} \frac{g_n(\mu)}{n}.$$ 

We conclude that $\sum_{k} \chi^E_+$ is an upper semicontinuous function as an infimum of a family of continuous functions. □

5. Fake foliations for partially hyperbolic dynamical systems

In this section we review some facts for invariant manifolds for partially hyperbolic systems. If $f \in \text{Diff}(M)$ is partially hyperbolic and $TM = E^s \oplus E^c \oplus E^u$, then the distributions $E^s$ and $E^u$ are uniquely integrable and the maximal integral manifolds of these distributions generate the stable and unstable manifolds respectively. In this case, the stable and unstable manifolds both foliate $M$. For a $C^r$ diffeomorphism the stable and unstable manifolds are both $C^r$ for each point in the manifold.

Unlike the stable and unstable manifolds the centerstable, centerunstable, and center manifolds are in general

- not $C^r$,
- not unique, and
- may not exist even locally.

To get around this we use a result from [10] that builds off of results from [20] to show that we have fake foliations around any point for the center direction. We will see that this will be sufficient for our results.

**Proposition 5.1.** (Proposition 3.1 in [10]) Let $f : M \to M$ be a $C^1$ partially hyperbolic diffeomorphism. For every $\epsilon > 0$, there exists constants $r > r_1 > 0$ such that for every $p \in M$, the neighborhood $B(p,r)$ is foliated by foliations $\hat{W}^u_p$, $\hat{W}^s_p$, $\hat{W}^c_p$, $\hat{W}^{cu}_p$, and $\hat{W}^{cs}_p$ with the following properties, for each $\beta \in \{u,s,c,cu,cs\}$.
Claim 5.2. \( \dot{\text{Almost tangency to invariant distributions.}} \) For each \( q \in B(p, r) \), the leaf \( \hat{W}_p^\beta(q) \) is \( C^1 \), and the tangent space \( T_q \hat{W}_p^\beta(q) \) lies in a cone of radius \( \epsilon \) about \( E^\beta(q) \).

(ii) \textbf{Local invariance.} For each \( q \in B(p, r_1) \),
\[ f(\hat{W}_p^\beta(q, r_1)) \subset \hat{W}_{f(q)}^\beta(f(q)) \] and
\[ f^{-1}(\hat{W}_p^\beta(q, r_1)) \subset \hat{W}_{f^{-1}(q)}^{\beta}(f^{-1}(q)). \]

(iii) \textbf{Exponential growth bounds at local scales.} The following hold for all \( n \geq 0 \):
1. Suppose that \( f^j(q) \in B(f^j(p), r_1) \) for \( 0 \leq j \leq n - 1 \). If \( q' \in \hat{W}_p^s(q, r_1) \), then \( f^n(q') \in \hat{W}_{f^n(p)}^s(f^n(q), r_1) \) and \( d(f^n(q), f^n(q')) \leq \lambda_0^nd(q, q') \). If \( f^j(q') \in \hat{W}_{f^j(p)}^s(f^j(q), r_1) \) for \( 0 \leq j \leq n - 1 \), then \( f^n(q') \in \hat{W}_{f^n(p)}^{cs}(f^n(q)) \) and \( d(f^n(q), f^n(q')) \leq \mu_0^n d(q, q') \).

(ii) Suppose that \( f^{-j}(q) \in B(f^{-j}(p), r_1) \) for \( 0 \leq j \leq n - 1 \). If \( q' \in \hat{W}_p^u(q, r_1) \), then \( f^{-n}(q') \in \hat{W}_{f^{-n}(p)}^u(f^{-n}(q), r_1) \) and \( d(f^{-n}(q), f^{-n}(q')) \leq \mu_0^n d(q, q') \). If \( f^{-j}(q') \in \hat{W}_{f^{-j}(p)}^{cs}(f^{-j}(q), r_1) \) for \( 0 \leq j \leq n - 1 \), then \( f^{-n}(q') \in \hat{W}_{f^{-n}(p)}^{cs}(f^{-n}(q)) \) and \( d(f^{-n}(q), f^{-n}(q')) \leq \lambda_0^{-n} d(q, q') \).

(iv) \textbf{Coherence.} \( \hat{W}_p^s \) and \( \hat{W}_p^c \) subfoliate \( \hat{W}_p^{cs} \), and \( \hat{W}_p^u \) and \( \hat{W}_p^{cu} \) subfoliate \( \hat{W}_p^{cu} \).

(v) \textbf{Regularity.} If \( f \) is \( C^2 \), then the foliations \( \hat{W}_p^u, \hat{W}_p^s, \hat{W}_p^c, \hat{W}_p^{cu}, \) and \( \hat{W}_p^{cs} \) are uniformly Hölder continuous. Furthermore, if \( \mu_0 \mu_1 < 1 \), and \( \lambda_0 \lambda_1 < 1 \), then the center leaves are \( C^2 \).

Using these foliations we get the following claim.

Claim 5.2. There exists \( \delta_0 > 0 \) such that if \( d(x, p) < \delta_0 \), then there exists a unique point \( y \in \hat{W}_p^{cs}(p) \) such that \( \hat{W}_p^u(x) \cap \hat{W}_p^{cs}(p) = \{ y \} \) and \( d(y, p) < Kd(x, p) < r_1 \).

Proof. The unique point of intersection follows from the foliations. The existence of \( \delta_0 \) and \( K \) follow from the fact that the foliations stay close to the bundles and these are uniformly transverse.

We now apply the above proposition to get the following lemma.

Lemma 5.3. If \( 0 < \delta < \min(\delta_0, r_1/K) \) where \( r_1 \) and \( K \) are as stated above, then for any \( p \in M \) and \( \hat{W}_p^{cs}(p) \) and \( \hat{W}_p^{cu}(p) \) fake centerstable and center manifolds respectively contained in sufficiently thin cone fields, then \( B^+(p, \infty, \delta) \subset \hat{W}_p^{cs}(p) \) and \( B(p, \infty, \delta) \subset \hat{W}_p^{cu}(p) \).

Proof. Let \( p \in M \), \( x \in B^+(p, \infty, \delta) \) and fix \( \hat{W}_p^\beta \) with \( \beta \in \{ u, s, c, cu, cs \} \) as in Proposition 5.1 where the foliations stay in sufficiently thin cone fields. Therefore there exists according to the above claim a unique point \( y \in \hat{W}_p^u(x) \cap \hat{W}_p^{cs}(p) \) with \( d(y, p) < r_1 \). As \( x \) belongs to the Bowen ball \( B^+(p, \infty, \delta) \) the point \( f^n(y) \) is the unique intersection \( \hat{W}_{f^n(p)}^u(f^n(x)) \cap \hat{W}_{f^n(p)}^{cs}(f^n(p)) \) for all \( n \geq 0 \) and \( y \in \)
If \( y \neq x \), then the uniform expansion and the exponential growth bounds in the fake center stable manifolds says that there exists some \( N \in \mathbb{N} \) such that \( d(f^N(p), f^N(y)) > r_1 \), a contradiction. Hence, \( y = x \).

Similarly, using backward iterates and the fact that \( \hat{W}^s(p) \) foliates \( \hat{W}^{cs}(p) \) we obtain that \( B(p, \infty, \delta) \subset \hat{W}^c(p) \).

6. Reparametrization of Bowen balls

In [7] the first author was able to show the existence of symbolic extensions for \( C^2 \) surface diffeomorphisms by a reparametrization of Bowen balls in a similar way to Yomdin’s theory [29] [30]. Let us first recall the settings.

Let \( B(0, r) \) be the open ball centered at the origin in \( \mathbb{R}^2 \) of radius \( r \in \mathbb{R}^+ \). We consider a sequence of \( C^2 \) maps \( T = (T_n)_{n \in \mathbb{N}} \) from \( B(0, 2) \subset \mathbb{R}^2 \) to \( \mathbb{R}^2 \). For all \( n \geq 1 \) we denote by \( T_n \) the composition \( T_n \circ ... \circ T_0 \) defined on the Bowen ball \( B_T(n, 2) := \{ x \in \mathbb{R}^2, T_k x \in B(0, 2) \text{ for all } k = 0, ..., n-1 \} \). We recall now the notion of finite time hyperbolic sets.

**Definition 6.1.** For any \( \chi^+, \chi^- > 0 > \gamma > 0 \) and \( C > 1 \), we denote for all integers \( n \) the set \( \mathcal{H}_T^n(\chi^+, \chi^-, \gamma, C) \) defined as

\[
\{ x \in B_T(n, 2) : \forall 1 \leq k \leq n, C^{-1} e^{(\chi^+ - \gamma)k} \leq \| D_x T^k \| \leq Ce^{(\chi^+ + \gamma)k} \}
\]

To bound the local entropy of an ergodic measure \( \nu \) we reparametrize Bowen balls intersected with finite time hyperbolic set associated to the parameters \( (\chi^+, \chi^-) = (\chi_1(\nu), \chi_2(\nu)) \) such that the map \( T^n \) composed with the reparametrizations are contracting. Then the local entropy of \( \nu \) is bounded by the logarithmic growth of the number of reparametrizations. By a combinatorial argument (see Lemma 7 of [7]) one can also prescribe the defect of multiplicativity of the norm of the composition \( DT_{i+1} \circ DT_i \) at each step \( 1 \leq i \leq n \), i.e. for all sequences of \( n \) positive integers \( K_n := (k_1, ..., k_n) \) we consider the subset \( \mathcal{H}_T(K_n) \) of \( B_T(n+1, 1) \) defined by the set of points \( y \in B_T(n+1, 1) \) such that

\[
\forall 1 \leq i \leq n, \left\lfloor \log \frac{\| D_y T^i \| \max(\| D_{T_y} T_{i+1} \|, 1)}{\| D_y T^{i+1} \|} \right\rfloor + 1 = k_i.
\]

We notice that for all \( y \in \mathcal{H}(K_n) \) we have

\begin{equation}
\sum_{i=1}^{n} k_i \leq n + \sum_{k=0}^{n} \log^+ \| D_{T_y} T_{i+1} \| - \log \| D_y T^{n+1} \|
\end{equation}
Typically the sequence $T = (T_n)_n$ is the local dynamics at one point of a $C^2$ dynamical system $(M, T)$ as defined below. Let us denote by $R_{inj}$ the radius of injectivity of $(M, ||||)$ and by $\exp : TM(R_{inj}) \to M$ the exponential map, where $TM(r) := \{(x, u), u \in T_xM, ||u|| < r\}$. We fix $R < R' < R_{inj}$ such that $T(B(x, R)) \subset B(Tx, R')$ for all $x \in M$. Let $x \in M$ and $n \in \mathbb{N}$. We consider the map $T^n_x : T_{T^{-1}x}M(R) \to T_{T^n x}M(R')$ defined by

$$T^n_x = \exp^{-1}_{T^n x} \circ T \circ \exp_{T^{-1}x}.$$

For all $\epsilon < R/2$, we put

$$T^n_{x,\epsilon} = \epsilon^{-1}T^n_x(\epsilon) : B(0, 2) \to T_{T^n x}M \simeq \mathbb{R}^2$$

and $T^n := (T^n_{x,\epsilon})_{n \in \mathbb{N}}$. By choosing $\epsilon > 0$ small enough one can assume the second derivative of $T^n_{x,\epsilon}$ with $k \geq 2$ is as small as we want, uniformly in $n$ and $x$.

We recall now the reparametrization lemma for $C^2$ surface maps proved in [7] which will be applied to the local dynamics in local center manifolds of a $C^2$ partially hyperbolic dynamical system with a two-dimensional central bundle.

Reparametrization Lemma 6.2. [7] Let $\chi^+ > 0 > \chi^-$ and $\frac{\min(\chi^+, \chi^-)}{3} > \gamma > 0$ and $C > 1$ and let $T := (T_n)_{n \in \mathbb{N}}$ be a sequence of $C^2$ diffeomorphisms onto their image from $B(0, 2)$ to $\mathbb{R}^2$ with $\sup_n ||DT_n|| < +\infty$ and $\sup_n ||DT_n^{-1}|| < +\infty$ and such that for all integers $n$ and for all $z, z' \in B(0, 2)$

$$||D^2T_n|| \leq \inf_{z \in B(0, 2)} ||D_zT_n||,$$

$$||D^2T_n^{-1}|| \leq \frac{\inf_{z \in B(0, 2)} ||(D_zT_n)^{-1}||}{\max(||DT_n||, 1)}.$$

Then there exists a real number $B$ depending only on $\chi^+, \chi^-, \gamma, C, \sup_n ||DT_n||$ and $\sup_n ||DT_n^{-1}||$ and an universal constant $A$ such that for all sequences $K_{n-1} = (k_1, ..., k_{n-1})$ of $n-1$ positive integers there exists a family $G_n$ of $C^1$ maps $\phi_n : [0, 1]^2 \to \mathcal{H}_2^T(\chi^+, \chi^-, \gamma, 2C)$ satisfying :

(i) $\forall \phi_n \in G_n$, $\phi_n([0, 1]^2) \subset B_T(n + 1, 2),$

(ii) $\forall \phi_n \in G_n, \forall k \leq n, ||D(T^k \circ \phi_n)|| \leq 1,$

(iii) $\mathcal{H}_2^T(\chi^+, \chi^-, \gamma, C) \cap \mathcal{H}_T(K_{n-1}) \cap B_T(n + 1, 1) \subset \bigcup_{\phi_n \in G_n} \phi_n([0, 1]^2),$ and

(iv) $\log \sharp G_n \leq B + An + 2 \sum_{i=1}^{n-1} k_i.$

7. Local Ruelle inequality for partially hyperbolic dynamical systems

Uniformly hyperbolic dynamical systems are expansive, i.e. the infinite (two sided) $\epsilon$-Bowen balls are reduced to singletons for $\epsilon$ small enough. In particular they are $h$-expansive and have zero tail entropy. On the other hand, measured entropy is bounded by the sum of positive Lyapunov exponents by Ruelle’s inequality [27]. For a partially hyperbolic dynamical system only the positive Lyapunov central exponents contribute to the creation of entropy at small scales.
Theorem 7.1. Let $M$ be a compact manifold and $f \in \text{Diff}^1(M)$ be partially hyperbolic. Let $(h_k)_k$ be an entropy structure of $f$. Then there exists an integer $k$ such that for all ergodic measures $\nu$ we have

$$(h - h_k)(\nu) \leq \sum \chi^c_k(\nu)$$

where $\sum \chi^c_k(\nu)$ denotes the sum of the positive Lyapunov exponents of $\mu$ in the center bundle. In particular, the tail entropy $h^s(f)$ satisfies

$$h^s(f) \leq \lim_{n \to +\infty} \frac{1}{n} \max_{k=1, \ldots, \dim(E_c)} \log^+ \|\Lambda^k DT^{n}\|_{E_c}.$$  

PROOF: By Lemma 5.3 there exists $\epsilon > 0$ such that the Bowen ball $B^\epsilon(x, \infty, \epsilon)$ is contained in the fake center manifold $\widehat{W}_c^\epsilon(x)$ for any $x \in M$. Let $\nu$ be an ergodic measure. We are going to show that

$$h^\text{New}(\nu, \epsilon) \leq \sum \chi^c(\nu).$$

We denote by $\psi_x : B(0, 2) \to M$ a $C^1$ embedding of the local center manifold at $x$ with $\psi_x(0) = x$. By compactness there exists $2 > R > 0$ such that

$$f \circ \psi_x(B(0, R)) \subset \psi_x(B(0, 2))$$

for all $x \in M$. We let $R > 1 > 0$ be such that $B(x, \epsilon) \cap \widehat{W}_c^\epsilon(x) \subset \psi_x(B(0, \epsilon'))$ and we consider the sequence of maps $F^x := (f^x_n)_{n \in \mathbb{N}}$ where $f^x_n : B(0, 2) \to \mathbb{R}^2$ is defined by

$$f^x_n = \epsilon'^{-1}\psi^{-1}_{f^{n-1}_x} \circ f \circ \psi_{f^{n-1}_x}(\epsilon').$$

We also denote by $f^{n,x} : B_{F_{x}}(n, 2) \to \mathbb{R}^2$ the composition

$$f^{n,x} := f^{n-1}_x \circ f^{n-2}_x \circ \ldots \circ f^{1}_x = \epsilon'^{-1}\psi^{-1}_{f^{n}_x} \circ f^{n} \circ \psi_{f^{n-1}_x}(\epsilon').$$

Since $\widehat{W}_c^\epsilon(x)$ lies in a thin cone field about $E^c(x)$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and for all $z \in B_{F_{x}}(n, 2)$ we have

$$\frac{1}{n} \log^+ \max_k \|\Lambda^k Dz f^{n,x}\|_k \simeq \frac{1}{n} \log^+ \max_k \|\Lambda^k D\psi_{x(\epsilon')} f^n\|_{E_c}.$$  

Fix $1 > \sigma > 0$ and let $F$ be a compact subset with $\nu(F) > \sigma$ and let $n_1 > n_0$ be an integer such that for all $y \in F$ we have

$$\frac{1}{n_1} \log^+ \max_k \|\Lambda^k D_y f^{n_1}\|_k \simeq \sum \chi^c_k(\nu).$$

and therefore for all $z \in B_{F_{x}}(n_1, 2) \cap \epsilon^{-1}\psi^{-1}_x(F)$

$$\frac{1}{n_1} \log^+ \max_k \|\Lambda^k D_z f^{n_1} f^{x}\|_k \simeq \sum \chi^c_k(\nu).$$

Let $\delta > 0$ be small enough such that the above equality also holds for any points $z$ which is $\delta$-close to $F$. Then for all $x$ the image by $f^{n_1,x}$ of any ball of radius
\( \delta \) centered at points of \( F \) may be covered by a family of balls of radius \( \delta \) whose logarithmic number does not exceed \( \sum \chi^+_c(\nu) \). Since typical \( \nu \)-points visit \( F \) with a frequency larger than \( 1 - \sigma \) we easily get

\[
\tau(B(x, \infty, \epsilon) \cap F, \delta) \leq (1 - \sigma)\text{Lip}(f) + \sum \chi^+_c(\nu)
\]

One easily concludes \( h^{\text{New}}(\nu, \epsilon) \leq \sum \chi^+_c(\nu) \) by taking the limit when \( \sigma \) goes to 1. \( \Box \)

**Remark 7.2.** Since entropy structures are invariant by taking the inverse by Lemma 3.1 there exists an integer \( k \) such that for all invariant measures \( \mu \) the local entropy \( (h - h_k)(\mu) \) is bounded from above both by the sums of the positive and the negative Lyapunov exponents. In particular, when the center bundle is two-dimensional any invariant measure \( \mu \) with \( (h - h_k)(\mu) > 0 \) is hyperbolic.

**8. Symbolic extension for partially hyperbolic dynamical systems with a 2-dimensional center bundle**

In this section we prove our main theorem and bound the symbolic extension entropy of a partially hyperbolic dynamical system with two dimension central subbundle by the entropy and the maximal Lyapunov exponent in the central direction. For \( \nu \) an ergodic measure in \( \mathcal{M}(M, f) \) let \( \chi^+_c(\nu) \) and \( -\chi^-_c(\nu) \) be the Lyapunov exponents in the center direction. If \( \mu \) is any measure in \( \mathcal{M}(M, f) \) we can extend this notion as follows:

\[
\overline{\chi}^+_c := \inf_{n \in \mathbb{N}} \frac{1}{n} \int_M \log^+ \|D_x f^n|_{E^c}\|d\mu(x)
\]

and

\[
\overline{\chi}^-_c := \inf_{n \in \mathbb{N}} \frac{1}{n} \int_M \log^+ \|D_x f^{-n}|_{E^c}\|d\mu(x).
\]

We notice that these are the harmonic extensions of the center Lyapunov functions for the ergodic measures.

**Theorem 8.1.** Let \( f : M \to M \) be a \( C^2 \) partially hyperbolic diffeomorphism satisfying the hypothesis for the main theorem. Then there exists a symbolic extension \( \pi : (Y, S) \to (M, f) \) such that

\[
h^\pi_{\text{ext}} - h = 2\overline{\chi}^+_c \quad (\text{resp.} \ 2\overline{\chi}^-_c)
\]

In particular, we have

\[
h_{\text{sex}} \leq h + 2 \min(\overline{\chi}^+_c, \overline{\chi}^-_c)
\]

and

\[
h_{\text{sex}}(f) \leq h_{\text{top}}(f) + 2 \limsup_{|n| \to +\infty} \frac{1}{|n|} \log^+ \|Df^n|_{E^c}\|.
\]

Thus \( f \) has symbolic extensions.
In relation to the above theorem there is the following conjecture which has been proved by T. Downarowicz and A. Maass in dimension one [17] and by the first author for $C^2$ surface diffeomorphisms [7] and nonuniformly entropy expanding maps [9].

**Conjecture 8.2.** Let $T : M \rightarrow M$ be a $C^r$ map with $r > 1$ on a compact manifold $M$ of dimension $d$. Then

$$h_{sex} \leq h + \frac{d \sum \chi_i^+}{r - 1}$$

where $\sum \chi_i^+$ denotes the sum of the positive Lyapunov exponents. In particular $T$ admits symbolic extensions.

We will prove that the center positive Lyapunov exponents $\chi_c^+$, in fact $2\chi_c^+$, is a fixed point of the monotone operator $T_{sex}$ (this exponent defines an affine upper semicontinuous function by Lemma 4.1). The above theorem follows then from Sex Theorem (Theorem 2.1). By Lemma 10.4.5 of [15] it is enough to estimate entropy structures in the closure of ergodic measures so that we are reduced to prove the following proposition (recall that $\left( \frac{1}{p} h_{New}^e(M|\cdot, \epsilon_k) \right)_k$ defines an entropy structure for all integers $p \neq 0$):

**Proposition 8.3.** Let $f : M \rightarrow M$ be a $C^2$ partially dynamical system with a 2-dimensional center bundle satisfying the hypothesis of the main theorem. Let $\mu$ be an $f$-invariant measure and fix some $\gamma > 0$. Then there exist $p_\mu \neq 0$, $\delta_\mu > 0$ and $\epsilon_\mu > 0$ such that for every ergodic measure $\nu$ with $\text{dist}(\nu, \mu) < \delta_\mu$ it holds that

$$\frac{1}{p_\mu} h_{New}^e(M|\nu, \epsilon_\mu) \leq 2 \chi_c^+(\mu) - 2 \chi_c^+(\nu) + \gamma. \quad (3)$$

**Proof:**

Observe that it is enough to consider ergodic measures with one positive and one negative Lyapunov exponent central exponent by the above local Ruelle inequality (Theorem 7.1). The proposition then follows as in [7] from the reparametrization lemma of Bowen balls. We replace Newhouse entropy by Modified Newhouse entropy and notice that the two-sided infinite Bowen balls are contained in central local manifolds. We present the main modifications and refer to [7] for details.

We do not work directly with $f$ but with an iterated $f^{p_\mu}$ such that the positive central Lyapunov exponents of $\mu$ are given by an almost additive process, that is

$$\chi_c^+(\mu) = \inf_n \frac{1}{n} \int_M \log^+ \|Df^n|_{E_c}\| d\mu(x) \simeq \frac{1}{p_\mu} \int_M \log^+ \|Df^{p_\mu}|_{E_c}\| d\mu(x)$$
As in the proof of Proposition 7.1 we denote by \( \psi_x : B(0, 2) \to M \) a \( C^2 \) embedding of the local center manifold at \( x \) with \( \psi_x(0) = x \). By compactness there exists \( 2 > R > 0 \) such that \( f^n \circ \psi_x(B(0, R)) \subseteq \psi_x(B(0, 2)) \) for all \( x \in M \). By Lemma 5.3 there exists \( \epsilon_0 > 0 \) such that the Bowen ball \( B^*(x, \infty, \epsilon_0) \) is contained in the fake center manifold \( \tilde{W}_x^c(x) \) for any \( x \in M \). For \( \epsilon < \epsilon_0 \) we let \( 0 < \epsilon' < R \) be such that \( B(x, \epsilon) \cap \tilde{W}_x^c(x) \subseteq \psi_x(B(0, \epsilon')) \) with \( \epsilon' \xrightarrow{\epsilon \to 0} 0 \) and we consider the sequence of maps \( F^n_\epsilon := (f_{n,\epsilon}^x)_n \) where \( f_{n,\epsilon}^x : B(0, 2) \to \mathbb{R}^2 \) is defined by \( f_{n,\epsilon}^x = \epsilon'^{-1} \psi \circ f^n \circ \psi^{-1} \). We also denote by \( f_{n,\epsilon}^{n,x} : B_{F^n}(n, 2) \to \mathbb{R}^2 \) the composition

\[
f_{n,\epsilon}^{n,x} := f_{n,\epsilon}^{n-1,x} \circ f_{n-1,\epsilon}^{n-2,x} \circ ... \circ f_{1,\epsilon}^{n-1,x} = \epsilon'^{-1} \psi^{-1} \circ f^{np} \circ \psi(f^{n-1}x).
\]

Since the \( C^2 \) norms of \( \psi_x \) and \( \psi^{-1}_x \) are uniformly bounded in \( x \) and the second derivative of \( f_{n,\epsilon}^x \) is going to zero uniformly in \( n \) and \( x \) when \( \epsilon \) goes to zero.

The parameters \( \epsilon_\mu \) and \( \delta_\mu \) are then chosen as follows:

- \( \epsilon_\mu < \epsilon_0 \) and the maps \( f_{n,\epsilon_\mu}^x \) have derivatives of order 2 satisfying the assumptions of the Reparametrization Lemma;
- \( \left| \int_M \log^+ \| D_y f^{pn} |_{E_c(y)} \| d\mu(y) - \int_M \log^+ \| D_y f^{pn} |_{E_c(y)} \| d\nu(y) \right| < < 1 \) for any ergodic measure \( \nu \) that is \( \delta_\mu \) close to \( \mu \).

We assume \( \nu \) to be ergodic for \( f^{pn} \) (if not one just considers an ergodic component of \( \nu \) and uses the harmonic property of the modified Newhouse entropy structure, see [7]). For any \( \sigma \in [0, 1] \) there exist a Borel set \( F \) with \( \nu(F) > \sigma \) such that for \( y \in F \)

\[
\frac{1}{p \mu^n} \log^+ \| D_y f^{pn} |_{E_c(y)} \|
\]

and

\[
\frac{1}{p \mu^n} \sum_{k=0}^{n-1} \log^+ \| D_y f^{pn} |_{E_c(f^{pk}y)} \|
\]

converge uniformly to \( \chi^+_c(\nu) \) and \( \frac{1}{p \mu} \int_M \log^+ \| D_y f^{pn} |_{E_c(y)} \| d\nu(y) \) when \( n \) goes to \(+\infty\). In particular, as \( \tilde{W}_x^c(x) \) lies in a thin cone field about \( E^c(x) \), there exists for any \( \gamma < < 1 \) a real number \( C > 1 \) such that

\[
\epsilon'^{-1} \psi^{-1}(F) \cap B_{F^n}(\infty, 2) \subseteq \bigcap_n \mathcal{H}_{F^n}(\chi^+_c(\nu), \chi^-_c(\nu), \gamma, C)
\]

By a combinatorial argument (Lemma 7 of [7]) one observes that the number of sets of the form \( \mathcal{H}_{F^n}(\mathcal{K}_n) \) having a non empty intersection with \( \mathcal{H}_{F^n}(\chi^+_c(\nu), \chi^-_c(\nu), \gamma, C) \) is exponentially small, see [7]. Then the Reparametrization Lemma allows us to reparametrize the intersection of \( F \) with Bowen balls in local center manifolds by maps which do not carry entropy (more precisely, they satisfy property (ii) of Lemma 6.2). Since infinite Bowen balls are contained in local center manifolds
the modified Newhouse entropy of $\nu$ is then bounded by the logarithmic number of reparametrization maps. Therefore we have according to Inequality (2) and Lemma 6.2 (iv):

$$\frac{1}{p_\mu} h^{\text{News}}_{f^{\mu}}(M|F, \epsilon_\mu) \leq 2 \limsup_n \frac{1}{p_\mu n} \sup_{x,z} \frac{1}{n} \left( \sum_{k=0}^{n-1} \log^+ \|D_{f^{\mu}_{p_\mu k,x}} f^{p_\mu}_{\epsilon_\mu} f^{p_\mu k x}\| - \log \|D_{z} f^{p_\mu n x}\| \right)$$

where the supremum holds over $x \in F$ and over $z \in \epsilon^{-1}_{\mu} \psi^{-1}_x(F) \cap B_{\epsilon_{\mu}}(\infty, 2)$.

We observe that for all $x$ and for all $z \in B_{\epsilon_{\mu}}(\infty, 2)$ we have

$$\frac{1}{p_\mu} \log^+ \|D_{z} f^{p_\mu n x}_{\epsilon_\mu}\| \approx \frac{1}{p_\mu} \log^+ \|D_{x} f^{p_\mu}|_{E_c(x)}\|$$

for $\epsilon_\mu$ small enough and for $p_\mu$ chosen large compared to the size of the derivative of $\psi_x$ and $\psi^{-1}_x$. Then we get together with (4):

$$\begin{align*}
\frac{1}{p_\mu} h^{\text{News}}_{f^{\mu}}(M|F, \epsilon_\mu) &\leq 2 \limsup_n \frac{1}{p_\mu n} \sup_{x \in F} \frac{1}{n} \left( \sum_{k=0}^{n-1} \log^+ \|D_{f^{\mu}_{p_\mu k,x}} f^{p_\mu}_{\epsilon_\mu} f^{p_\mu k x}\| |_{E_c(f^{\mu} x)}\| - \chi_c^+(\nu) \right) \\
& \leq 2 \left( \frac{1}{p_\mu} \int_M \log^+ \|D_{z} T^{p_\mu n}|_{E_c(y)}\| d\nu(x) - \chi_c^+(\nu) \right) \\
& \leq 2 \left( \chi_c^+(\mu) - \chi_c^+(\nu) \right)
\end{align*}$$

The proposition follows by letting $\sigma$ go to 1.  

\[ \square \]

REFERENCES


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