NONLOCALLY MAXIMAL AND PREMAXIMAL
HYPERBOLIC SETS

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Abstract. We prove that for any closed manifold of dimension 3 or greater there is an open set of smooth flows that have a hyperbolic set that is not contained in a locally maximal one. Additionally, we show that the stabilization of the shadowing closure of a hyperbolic set is an intrinsic property for premaximality. Lastly, we review some results due to Anosov that concern premaximality.

1. Introduction

Since the 1960s the study of hyperbolic sets has been a cornerstone in the field of dynamical systems. These sets are remarkable not only in their complexity, but also in the fact that they persist under perturbations. Additionally, for a point in a hyperbolic set the derivative of the map at this point gives information on the local dynamics for the original nonlinear map.

As a reminder, for a diffeomorphism \( f : M \to M \), a compact invariant set \( \Lambda \) is hyperbolic for \( f \) if \( T_\Lambda M = E^s \oplus E^u \) is a \( Df \)-invariant splitting such that \( E^s \) is uniformly contracted and \( E^u \) is uniformly expanded by \( Df \).

Anosov was one of the pioneers in studying hyperbolic sets. Indeed, if the entire manifold is a hyperbolic set for a diffeomorphism, then the diffeomorphism is called Anosov. This is one of the best understood classes of hyperbolic sets.

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Another important class of hyperbolic sets are those that are locally maximal. For a compact metric space $X$ and a continuous homeomorphism $T : X \to X$ a set $K \subset X$ is \textit{locally maximal} if there exists a neighborhood $U$ of $K$ such that

$$K = \bigcap_{n \in \mathbb{Z}} T^n(U).$$

Hence, such sets are the maximal invariant set within $U$.

Locally maximal sets were defined more or less simultaneously by Anosov and Conley (who called these sets \textit{isolated}) in the 1960s. Alexseev proved in [1] that a shift space is locally maximal if and only if it is a shift of finite type. Furthermore, he proved that given any shift space $\Sigma$ and a neighborhood $U$ of $\Sigma$ there exists a locally maximal shift $\Sigma'$ containing $\Sigma$ and contained in $U$.

A question that was posed in the 1960s by Anosov, Alexseev, and others (that is stated for instance in [15, p. 272]), is the following:

**Question 1.1.** \textit{If $\Lambda$ is a hyperbolic set and $U$ is a neighborhood of $\Lambda$, then is there a locally maximal hyperbolic set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset U$?}

As stated by Anosov in [3] it was hoped at the time that the answer would be in the affirmative.

One reason this was hoped for is that locally maximal hyperbolic sets are more easily classified. Indeed, a hyperbolic set is known to be locally maximal if and only if it has a local product structure (defined in Section 2). A standard assumption used in characterizing topological and/or measure theoretic properties of hyperbolic sets is that the set is locally maximal. For instance, Smale’s Spectral Decomposition Theorem, see for instance [15, p. 575], is valid for locally maximal hyperbolic sets. In fact, quite frequently even when this is not specifically stated in a theorem one finds in the proof that this assumption is required for the result to hold.

1.1. \textbf{Nonlocally maximal hyperbolic sets.} It was shown by Crovisier in [9] that there is a hyperbolic set on the 4-torus that is never included in a locally maximal set. Later, in [13] it was shown that any compact boundaryless manifold with dimension greater than or equal to 2 has a $C^r$ open set of diffeomorphisms where $1 \leq r \leq \infty$ such that each diffeomorphism in the open set contains a hyperbolic set that is not included in a locally maximal one.

Our first result is an extension of the results in [13] to hyperbolic flows. As a reminder, for a smooth flow $\phi : \mathbb{R} \times M \to M$, a compact $\phi$-invariant set $\Lambda$ is hyperbolic for $\phi$ if $T\Lambda M = E^s \oplus E^c \oplus E^u$ is a flow invariant splitting such that $E^s$ is uniformly contracted, $E^c$ is the flow
direction, and $E^u$ is uniformly expanded. A set $\Lambda$ is locally maximal for the flow $\phi$ if there is an open set $U$ containing $\Lambda$ such that

$$\Lambda = \cap_{t \in \mathbb{R}} \phi_t(U).$$

**Theorem 1.2.** Let $M$ be a compact, boundaryless $C^r$ manifold for $1 \leq r \leq \infty$ with $\dim M \geq 3$ and $\mathcal{X}^k(M)$ be the set of $C^k$ flows on $M$ where $1 \leq k \leq r$. Then there exists a $C^k$ open set of flows on $M$ such that each flow contains a hyperbolic set not contained in a locally maximal one.

**Remark 1.3.** We notice the following.

1. If $\dim M \leq 2$ then every hyperbolic set for a smooth flow is a finite union of hyperbolic closed trajectories and hence it is locally maximal.

2. Also, one can always suspend the map constructed in [13] and have a topological flow. However, it remains to be seen if the suspension will still be smooth. Also, the suspension could introduce nontrivial topology and it may not be possible to obtain the result of 1.2 on any manifold of dimension 3 or larger.

As in [13, Theorem 1.5] we can show the following result.

**Theorem 1.4.** Let $\Lambda$ be a hyperbolic set for a flow and $U$ be a neighborhood of $\Lambda$, then there exists a hyperbolic set $\Lambda'$ with a Markov partition for the flow such that $\Lambda \subset \Lambda' \subset U$.

The proof is very similar to that in [13]. Indeed, the necessary theorems used in [13] to prove the similar results for maps hold for hyperbolic sets for flows, and hence the proof is left to the reader.

The hyperbolic set constructed in Theorem 1.2 above need not transitive under the flow. However, as in [13, Theorem 1.6] one can construct a flow in higher dimensions with a transitive hyperbolic set that is not contained in a locally maximal one.

1.2. **Premaximality.** In this paper we also examine conditions under which a hyperbolic set, $\Lambda$, is included in a locally maximal hyperbolic set within an arbitrarily small neighborhood of $\Lambda$.

Following the terminology introduced by Anosov in [3] we define a hyperbolic set $\Lambda$ for a diffeomorphism to be **premaximal** if for any open set $U$ containing $\Lambda$ there is a locally maximal hyperbolic set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset U$. In [3] Anosov proves that any zero-dimensional hyperbolic set for a diffeomorphism is premaximal, and in [2] Anosov proves there is an intrinsic property for premaximal hyperbolic sets for diffeomorphisms. Moreover the following holds.
Theorem 1.5. [6] Let \( f : M \to M \) and \( f' : M' \to M' \) be diffeomorphisms, \( \Lambda \) a hyperbolic set for \( f \), \( \Lambda' \) a hyperbolic set for \( f' \), and \( h : \Lambda \to \Lambda' \) a homeomorphism such that \( h \circ f = f' \circ h \). If \( U \) is a neighborhood of \( \Lambda \) and \( U' \) is a neighborhood of \( \Lambda' \), then there exists neighborhoods \( V \subset U \) of \( \Lambda \) and \( V' \subset U' \) of \( \Lambda' \) and continuous injective equivariant maps \( h_1 : I_f(U) \to M' \) and \( h_2 : I_{f'}(U') \to M \) such that \( h_1|_\Lambda = h \), and

\[
\begin{align*}
    h_1(I_f(V)) &\subset I_{f'}(U'), & h_2(I_{f'}(V')) &\subset I_f(U), \\
    h_1 \circ f|_{I_f(V)} &= g \circ h_1|_{I_f(V)}, & f \circ h_2|_{I_{f'}(V')} &= h_2 \circ f'|_{I_{f'}(V')}, \\
    h_2 \circ h_1|_{I_f(V)} &= \text{id}, \quad \text{and} & h_1 \circ h_2|_{I_{f'}(V')} &= \text{id}.
\end{align*}
\]

The above theorem shows that \( f|_\Lambda \) defines the set of trajectories that lie in a sufficiently small neighborhood of \( \Lambda \). However, in [6] the specific intrinsic property for premaximality is not stated.

We extend result of [6] to the case of flows and prove the premaximality is an intrinsic property for hyperbolic sets for flows.

Let \( X \) and \( X' \) be vector fields on smooth compact Riemannian manifolds \( M \) and \( M' \) respectively. Denote by \( \phi \) and \( \phi' \) flows generated by them. An increasing homeomorphism of the real line \( \alpha : \mathbb{R} \to \mathbb{R} \) is called a reparametrization. Let \( \Lambda \) and \( \Lambda' \) be hyperbolic sets for \( X \) and \( X' \) respectively. We say that \( \Lambda \) and \( \Lambda' \) are topologically equivalent if there exists a homeomorphism \( h : \Lambda \to \Lambda' \) and a continuous map \( \alpha : M \times \mathbb{R} \to \mathbb{R} \) such that

\[
    h \circ \phi_t(x) = \phi'(\alpha(x, t), h(x)), \quad x \in \Lambda, t \in \mathbb{R}
\]

where \( \alpha(x, \cdot) \) is a reparametrization for each \( x \in \Lambda \). In this case there exists a continuous map \( \beta : M' \times \mathbb{R} \to \mathbb{R} \), such that

\[
    \beta(h(x), \alpha(x, t)) = t, \quad x \in \Lambda, t \in \mathbb{R}
\]

\[
    \alpha(h^{-1}(x'), \beta(x', t')) = t', \quad x' \in \Lambda', t' \in \mathbb{R}
\]

Theorem 1.6. Let \( \Lambda \) and \( \Lambda' \) be hyperbolic sets for vector fields \( X \) and \( X' \) respectively. Assume that \( \Lambda \) and \( \Lambda' \) are topologically equivalent. Then \( \Lambda \) is premaximal if and only if \( \Lambda' \) is premaximal.

Below we provide an equivalent condition to premaximality for diffeomorphisms and flows. Before stating the result we define some important terms involving shadowing. Let \( \Phi = \phi(t, x) \) be a dynamical system where \( t \) can be taken to be discrete or continuous. If \( t \) is discrete, we assume the dynamical system is generated by a diffeomorphism of a compact manifold to itself. If \( t \) is continuous, we assume that the dynamical system is generated by a smooth vector field on a compact
manifold. Let $a > 0$ be an expansivity constant for some neighborhood of a hyperbolic set $\Lambda$ and let $\delta_0 > 0$ be such that any $\delta_0$-pseudo orbit in $\Lambda$ can be $a/2$-shadowed by an exact trajectory (definitions are given in the next section). Note that due to expansivity the shadowing trajectory is unique (for the case of a flow this is true up to a reparametrization).

For $\Lambda$ a hyperbolic set for a diffeomorphism and $\delta \in (0, \delta_0)$ the shadowing closure (or $\delta$-shadowing closure) of $\Lambda$ is

$$\text{sh}(\Lambda, \delta) = \{ y \in M : y \text{ shadows a } \delta\text{-pseudo orbit in } \Lambda \}.$$ 

For a fixed $\delta > 0$ we can construct a sequence of shadowing closures $\Lambda_0, \Lambda_1, \ldots$, where $\Lambda_0 = \Lambda$ and $\Lambda_j = \text{sh}(\Lambda_{j-1}, \delta)$ for $j \in \mathbb{N}$. We say a shadowing sequence stabilizes if $\Lambda_j = \Lambda_{j+1}$ for all $j \geq N$ where $N \in \mathbb{N}$.

**Theorem 1.7.** For a hyperbolic set $\Lambda$ of a dynamical system $\Phi$ the following statements are equivalent

1. $\Lambda$ is premaximal;
2. for any neighborhood $U$ of $\Lambda$ the shadowing closure stabilizes inside $U$ for some $\delta > 0$.

Note that due to Theorem 1.5 and Theorem 1.6 the second property in Theorem 1.7 for diffeomorphisms is intrinsic.

The paper proceeds as follows. In Section 2 we review relevant background on hyperbolicity and flows. In Section 3 we prove Theorem 1.2. In Section 4 we review the results of Anosov in [2, 3, 4, 5, 6] and prove Theorems 1.6 and 1.7.

### 2. Background

#### 2.1. Hyperbolic sets for flows.

We first review properties of hyperbolic sets for flows.

**Definition 2.1.** Let $X$ be a metric space and $\phi$ a continuous flow on $X$. Then for $x \in X$ we define the stable set

$$W^s(x) := \{ y \in X : \lim_{t \to \infty} d(\phi_t(x), \phi_t(y)) = 0 \}.$$ 

Further, for $\varepsilon > 0$ the $\varepsilon$-stable set is

$$W^s_\varepsilon(x) := \{ y \in W^s(x) : d(\phi_t(x), \phi_t(y)) \leq \varepsilon \text{ for all } t \geq 0 \}.$$ 

Note that the unstable sets $W^u(x)$ and $W^u_\varepsilon(x)$ are defined identically under the flow $\phi^{-t}$. Furthermore, we define the center-stable set

$$W^{cs}(x) := \{ \phi_t(W^s(y)) \}_{t \in \mathbb{R}} = \bigcup_{y \in \phi_t(x)} W^s(y).$$
The center-unstable set of \( x \) is defined to be the center-stable set of \( x \) under \( \phi_{-t} \). We will also use the notation \( W_{\text{loc}}^s \) to mean \( W_{\varepsilon}^s \) for sufficiently small \( \varepsilon \) (depending on the context) and \( W_{\text{loc}}^u \) similarly to mean \( W_{\varepsilon}^u \) for small \( \varepsilon \).

Let \( X \) be a manifold and \( \phi \) be a \( C^r \) flow on \( X \). If \( \Lambda \) is a hyperbolic set for \( \phi \) and \( p \in \Lambda \), then \( W_{\varepsilon}^s(p) \) is a \( C^r \) immersed submanifold and is an immersed copy of \( \mathbb{R}^k \) where \( k = \text{dim } E^s(x) \). Similar statements hold for the unstable sets, center-stable sets, and center-unstable sets. Also note that the stable and unstable manifolds vary continuously on the point \( p \).

**Definition 2.2.** For a metric space \( X \) and a flow \( \phi \), a set \( \Gamma \subset X \) is said to have a **local product structure** if for all \( \varepsilon > 0 \) sufficiently small there exists a \( \delta > 0 \) such that given \( x, y \in \Gamma \) with \( d(x, y) < \delta \) we have, for some real \( |t| < \varepsilon \), a unique point \( S(x, y) := b \in W_{\varepsilon}^u(\phi_t(x)) \cap W_{\varepsilon}^s(y) \) contained in \( \Gamma \).

**Remark 2.3.** Note that for any hyperbolic set \( \Lambda \) there always exist constants \( \delta \) and \( \varepsilon \) sufficiently small such that \( x, y \in \Lambda \) and \( d(x, y) < \delta \) implies \( S(x, y) = W_{\varepsilon}^u(\phi_t(x)) \cap W_{\varepsilon}^s(y) \) is a unique point in the manifold, but may not be in \( \Lambda \).

The following lemma is also critical to the paper. Note that this lemma is almost always stated and proved for maps, but is in fact true for flows as well (see [8] and [16]).

**Lemma 2.4.** A hyperbolic set \( \Gamma \) has a local product structure if and only if it is locally maximal.

We also need the notion of the shadowing property. For \( \delta > 0 \) a map \( g : \mathbb{R} \rightarrow M \) is an \( \delta \)-pseudo orbit if the following holds

\[
d(g(t + \tau), \phi_\tau(g(t))) < \delta, \quad t \in \mathbb{R}, |\tau| < 1.
\]

A \( \delta \)-pseudo orbit \( g \) is \( \varepsilon \)-shadowed by a point \( x_0 \) if there exists a reparametrization \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) satisfying

\[
d(g(t), \phi_{\alpha(t)}(x_0)) < \varepsilon, \quad \text{and}
\]

\[
\left| \frac{\alpha(t_1) - \alpha(t_2)}{t_1 - t_2} - 1 \right| < \varepsilon \quad \text{for } t_1 \neq t_2.
\]

A vector field \( X \) is expansive on a compact metric space \( W \) if there exist constants \( a, \tau_0 > 0 \) such that if \( x_1, x_2 \in W \) and there exists a reparametrization \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) such that the following inequalities hold

\[
d(\phi(\alpha(t), x_1), \phi(t, x_2)) < a, \quad t \in \mathbb{R},
\]

then \( x_2 = \phi(\tau, x_1) \), where \( \tau \in (-\tau_0, \tau_0) \).
Theorem 2.5. Let $\Lambda$ be a hyperbolic set for a flow $\phi$ on a compact manifold $M$. Then the following hold:

- there exists a neighborhood $W$ of $\Lambda$ such that $\phi$ is expansive on
  $$I_\phi(W) := \bigcap_{t \in \mathbb{R}} \phi_t(W);$$

  and

- there exists a neighborhood $U(\Lambda) \subset W$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that any $\delta$-pseudo orbit $g \subset U$ can be $\varepsilon$-shadowed by some point $x_0 \in M$.

Definition 2.6. For $\phi$ a flow on a compact metric space $X$ a nonempty subset $A$ of $X$ is called an attractor if it satisfies the following three conditions.

(i) $A$ is forward-invariant under $\phi$; i.e., $x \in A$ implies $\phi_t(x) \in A$ for all $t > 0$.

(ii) There exists a neighborhood of $A$, called the basin of attraction of $A$ and denoted $\mathcal{B}(A)$, which consists of all points that tend towards $A$ under $\phi_t$ as $t \to \infty$. In other words, $\mathcal{B}(A)$ consists of all points $x$ such that for any open neighborhood $N$ of $A$, there exists $T > 0$ such that $\phi_t(x) \in N$ for all $t > T$.

(iii) No proper subset of $A$ satisfies conditions (i) and (ii).

When an attractor $\Lambda$ is (uniformly) hyperbolic we know that

- periodic points are dense in $\Lambda$,
- $x \in \Lambda$ implies $W^{cu}(x) \subset \Lambda$, and
- for any periodic point $x \in \Lambda$ we know $\bigcup_{y \in \mathcal{O}(x)} W^{cs}(y)$ is dense in $\mathcal{B}(\Lambda)$.

We will need the following technical result, known in the literature as the Inclination Lemma, or $\lambda$-lemma. The statement can be found in [7]. Note that the statement for hyperbolic periodic points would be similar.

Lemma 2.7 (Inclination Lemma). Let $p \in M$ be a hyperbolic fixed point for a $C^r$ flow $\phi$, for $r \geq 1$, with local stable and unstable manifolds $W^{s}_{loc}(p)$ and $W^{u}_{loc}(p)$, respectively. Fix an embedded disk $B$ in $W^{u}_{loc}(p)$ which is a neighborhood of $p$ in $W^{u}_{loc}(p)$, and fix a neighborhood $V$ of this disk in $M$. Let $D$ be a transverse disk to $W^{s}_{loc}(p)$ at a point $z$ such that $D$ and $B$ have the same dimension. Write $D_t$ for the connected component of $\phi_t(D) \cap V$ which contains $\phi_t(z)$, for $t \geq 0$. Then, given $\varepsilon > 0$ there exists $T > 0$ such that for all $t > T$ the disk $D_t$ is $\varepsilon$-close to $B$ in the $C^r$-topology.
2.2. Hyperbolic sets of diffeomorphisms. Since the statement of Theorem 1.7 refers to diffeomorphisms as well as smooth flows we now review some definitions for discrete dynamical systems.

For \( f : X \to X \) a homeomorphism of a metric space and \( \delta > 0 \) an \( \delta \)-pseudo orbit is a sequence \( \{x_j\}_m^l \) where

- \( l \in \{-\infty\} \cup \mathbb{Z}, \ m \in \mathbb{Z} \cup \{\infty\}, l < m, \) and
- \( d(f(x_j), x_{j+1}) < \delta \) for all \( j \in [l, m] \).

For a \( \delta \)-pseudo orbit \( \{x_j\}_m^l \) we say this sequence is \( \varepsilon \)-shadowed by a point \( x \in X \) if \( d(f^j(x), x_j) < \varepsilon \) for \( j \in [l, m] \).

We say that a homeomorphism \( f \) of a compact metric space \( W \) is expansive if there exists a constant \( a > 0 \) such that if \( x_1, x_2 \in I_f(W) := \bigcap_{n \in \mathbb{Z}} f^n(W) \) and \( d(f^n(x_1), f^n(x_2)) < a, \ n \in \mathbb{R} \), then \( x_2 = x_1 \).

**Theorem 2.8. (Shadowing Lemma)** Let \( \Lambda \) be a hyperbolic set for \( f : M \to M \) a diffeomorphism. Then there exists neighborhood \( U(\Lambda) \) such that for all \( \varepsilon > 0 \) there exists an \( \delta > 0 \) such that if \( \{x_j\}_{-\infty}^\infty \subset U \) is an \( \delta \)-pseudo orbit, then there exists \( x \in M \) that \( \varepsilon \)-shadows \( \{x_j\}_{-\infty}^\infty \).

Let \( f : M \to M \) be a diffeomorphism and \( \Lambda \) be a hyperbolic set for \( f \). For \( \varepsilon > 0 \) sufficiently small and \( x \in \Lambda \) the local stable and unstable manifolds and stable and unstable manifolds are defined similar to the case for flows and for a \( C^r \) diffeomorphism \( f \) the stable and unstable manifolds of a hyperbolic set are \( C^r \) injectively immersed submanifolds.

For \( \Lambda \) a hyperbolic set we know that if \( \varepsilon \) is sufficiently small and \( x, y \in \Lambda \), then \( W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \) consists of at most one point. For such an \( \varepsilon > 0 \) define

\[
D_\varepsilon = \{(x, y) \in \Lambda \times \Lambda \mid W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \in \Lambda\}
\]

and \([\cdot, \cdot] : D_\varepsilon \to \Lambda \) so that \([x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)\).

We will also need openness of hyperbolicity.

**Lemma 2.9.** Let \( \Lambda \subset M \) be a hyperbolic set of the diffeomorphism \( f : U \to M \). Then for any open neighborhood \( V \subset U \) of \( \Lambda \) and every \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that if \( f' : U \to M \) and \( d_{C^1}(f|_V, f') < \varepsilon \) there is a hyperbolic set \( \Lambda' = f'(\Lambda') \subset V \) for \( f' \) and a homeomorphism \( h : \Lambda' \to \Lambda \) with \( d_{C^0}(Id, h) + d_{C^0}(Id, h^{-1}) < \delta \) such that \( h \circ f'|_{\Lambda'} = f|_{\Lambda'} \circ h \).

Moreover, \( h \) is unique when \( \delta \) is sufficiently small.
2.3. **Normal hyperbolicity.** For embedding constructions into higher dimensions, we will need the notion of normal hyperbolicity. A normally hyperbolic invariant manifold (NHIM) is a generalization of a hyperbolic fixed point and a hyperbolic set. Fenichel proved that NHIMs and their stable and unstable manifolds are persistent under perturbation [11], [12]. We define NHIMs for maps, but the definition for flows is similar (and more technical).

**Definition 2.10.** Let $M$ be a compact smooth manifold and $f : M \to M$ a diffeomorphism. Then an $f$-invariant submanifold $\Lambda$ of $M$ is said to be a **normally hyperbolic invariant manifold** if there exist $m \in \mathbb{N}$ such that the mapping $f^m$ satisfies the following property. There exists a continuous invariant bundle $T_\Lambda M = E^s \oplus T\Lambda \oplus E^u(x)$, and continuous positive functions $\nu, \hat{\nu}, \gamma, \hat{\gamma} : M \to \mathbb{R}$ such that

- $\nu, \hat{\nu} < 1$, $\nu < \gamma < \hat{\gamma} < \hat{\nu}^{-1}$
- for all $x \in \Lambda$, $v \in T_x M$, $|v| = 1$
- $|Df^m(x)v| \leq \nu(x)$, $v \in E^s(x)$;
- $\gamma(x) \leq |Df^m(x)v| \leq \hat{\gamma}(x)$, $v \in T\Lambda$;
- $|Df^m(x)v| \geq \hat{\nu}^{-1}(x)$, $v \in E^u(x)$.

Adapting the above for flows gives us an important result ([10, p. 215]) which says that if a $C^r$ vector field $Y$ in some $C^1$ neighborhood of our original vector field $X$ (equated with a flow $\phi$, under which $\Lambda$ is invariant) there is a $C^r$ manifold $\Lambda_Y$ invariant under $Y$ and $C^r$ diffeomorphic to $\Lambda$. An immediate consequence of this is that the dynamics on $\Lambda_Y$ under the vector field $Y$ are a perturbation of the dynamics of $\Lambda$ under $X$.

### 3. Nonlocally maximal sets for flows

The foundation of the proof of Theorem 1.2 is the Plykin attractor, see for instance [15, p. 537-41] for a construction of the Plykin attractor. The first author used this map with some modifications to prove Theorem 1.3 in [13] on the existence of hyperbolic sets not included in locally maximal ones. To extend the results of [13] to flows we need to show the suspension can be done smoothly and in such a way that no nontrivial topology is introduced in the suspension.

In [14] Hunt shows that the Plykin attractor can be suspended smoothly in such a way that the suspended flow is on the solid 2-torus. The result of the construction is a smooth flow of a solid 2-torus where the basin of attraction for the suspended Plykin attractor is the
Proof of Theorem 1.2. We now show how to adapt the construction in [13] to the suspension flow described above. First we would like to sketch the main steps of the construction of the example for a more detailed proof see [17].

Let $T_1$ be the solid 2-torus used in the construction described above. We first embed Hunt’s Plykin attractor in a slightly larger solid 2-torus, $T$, so that the flow is the identity in a neighborhood of the boundary and then extend the flow to a 3-disk, $D$, so that it is the identity on $D - T$. This will allow us to embed the example into any arbitrary 3-manifold.

We now modify the flow on $T_1$ to incorporate the construction in [13]. Let $\Lambda_a$ be the Plykin attractor and $\phi$ be the flow on $D$. Fix some $p \in \partial T_1$. Take a sufficiently small open neighborhood $U$ of $O(p)$, small enough to be disjoint from $\partial T$ and $\Lambda_a$, and alter $\phi$ in $U$ so that $p$ is a hyperbolic saddle periodic point with $W^s(p) \subset U \cap \partial T_1$. Also, $W^{cu}(p) \setminus O(p)$ contains two components. One of which is contained in $T - T_1$ and the other, denoted $W^s(p)$ is contained strictly in the interior of $T_1$.

Let $q$ be a periodic point in $\Lambda_a$. Since $W^{cs}(q) = W^s(\Lambda_a) = \text{int}(T_1)$ we know that given any point of $W^s(p)$ that there must exist some point in $W^{cs}(q)$ arbitrarily close to it. Fix $z \in W^s(p)$. Perturb the flow in a neighborhood of $z$ so that $z \in W^{cs}(q) \cap W^s(p)$. This can be done since $z$ is a wandering point for the flow.

Here we will need two definitions. A hyperbolic set $\Lambda$ for a $C^1$ flow has a heteroclinic tangency if there exist $x, y \in \Lambda$ such that $W^s(x) \cap W^u(y)$ contains a point of tangency. A point of quadratic tangency for a $C^2$ flow is defined as a point of heteroclinic tangency where the curvature of the stable and unstable manifolds differs at the point of tangency.

Now after a further perturbation to the flow as in [13] there exists a point $w \in W^u(z)$ and a point $q' \in W^u(q)$ such that $W^u(z)$ and $W^s(q')$ have a quadratic tangency at $w$. Let $I$ be the segment of $W^u(p)$ from $z$ to $w$, and let $J$ be the segment of $W^u(q)$ from $q$ to $q'$. The resulting flow will contain a hyperbolic set that cannot be contained in a locally maximal set as we show below. Figure 1 demonstrates a cross section of the constructed flow.

Let $\Lambda = \Lambda_a \cup O(p) \cup O(z)$. Standard arguments as in [13], adapted to flows, show that this is a hyperbolic set under $\phi$. Now suppose
Λ ⊂ Λ', where Λ' is a locally maximal hyperbolic set. Modifying the proof in [13] we see from the product structure on Λ' that this implies \( w \in Λ' \), a contradiction. Hence, Λ is not contained in a locally maximal hyperbolic set.

We now show the construction is robust under perturbation. Since transversality is trivially open, and hyperbolicity is open by Lemma 2.9, it is sufficient to show that there remains a point \( \tilde{w} \in W^u(\tilde{p}) \cap W^s(\tilde{u}) \) for some \( \tilde{u} \in W^{cu}_{loc}(\tilde{q}) \). Let \( \tilde{p} \) and \( \tilde{q} \) be the continuations of \( p \) and \( q \) for the perturbed flow. By construction, the stable manifolds for all the \( x \in W^{cu}_{loc}(\tilde{q}) \) locally foliate the region, so there must exist a point \( \tilde{u} \in W^{cu}_{loc}(\tilde{q}) \) and a point \( \tilde{w} \in W^{cs}(\tilde{u}) \cap W^u(\tilde{p}) \) such that the one-dimensional path \( W^u(\tilde{p}) \) remains tangent to the two-dimensional plane \( W^{cs}(\tilde{u}) \) at \( \tilde{w} \). Specifically, we have \( T_\tilde{w}W^u(\tilde{p}) \subseteq T_\tilde{w}W^{cs}(\tilde{u}) \).

Using normal hyperbolicity we can embed our example into any smooth manifold \( M \) of dimension greater than 3. Furthermore, normal hyperbolicity implies the construction is still robust in this setting. □

4. Premaximality

Before we proceed to the proof of Theorems 1.6 and 1.7 let us review results by Anosov on premaximality [2, 3, 4, 5, 6]. As was mentioned in Section 1 Theorem 1.5 implies that premaximality is an intrinsic property.
Let us recall the main idea of the proof of Theorem 1.5. For \( \delta > 0 \) denote \( P(\delta) \) as the set of \( \delta \)-pseudo orbits \( \{y_n \in \Lambda\}_{n \in \mathbb{Z}} \) endowed with the Tikhonov product topology. Consider the shift map \( \sigma : P(\delta) \rightarrow P(\delta) \) defined as \( \sigma(\{y_n\}) = \{y_{n+1}\} \). The Shadowing Lemma implies that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \{y_n\} \in P(\delta) \) there exists point \( x \) such that

\[
(1) \quad d(y_n, f^n(x)) < \varepsilon.
\]

Note that due to the expansivity property we know for small enough \( \varepsilon > 0 \) that such a point \( x \) is unique. Fix such an \( \varepsilon > 0 \) and a corresponding \( \delta \) from the shadowing lemma. Consider the map \( T : P(\delta) \rightarrow M \) defined by the condition that for \( \{y_n\} \in P(\delta) \) the point \( x = T(\{y_n\}) \) is the unique point satisfying \( (1) \). It is easy to show that

\[
(2) \quad T \circ \sigma = f \circ T.
\]

Now let us consider \( c > 0 \) and a small neighborhood \( U \subset B(c, \Lambda) \) of \( \Lambda \) and a point \( z \) such that \( O(z) \subset U \). There exists a sequence of points \( \{y_n\} \) satisfying

\[
(3) \quad d(y_n, f^n(z)) < c.
\]

For any \( \delta > 0 \) there exists a \( c > 0 \) such that inequality \( (3) \) implies \( \{y_n\} \in P(\delta) \) and

\[
(4) \quad T(\{y_n\}) = z.
\]

We can similarly define for \( \Lambda' \) and \( f' \) sets \( P'(\delta) \) and maps \( T', \sigma' \). Note that similarly to \( (2) \) the equality

\[
(5) \quad T' \circ \sigma' = f' \circ T'
\]

holds. For any \( \delta'_1 > 0 \) there exists \( \delta_1 > 0 \) such that

\[
(6) \quad h(P(\delta_1)) \subset P'(\delta'_1).
\]

(Recall that \( h : \Lambda \rightarrow \Lambda' \) is a conjugacy between \( \Lambda \) and \( \Lambda' \).) Similarly for any \( \delta_2 \) there exists \( \delta'_2 \) such that

\[
(7) \quad h^{-1}(P'(\delta'_2)) \subset P(\delta_2).
\]

Equations \( (2), (4), (5), (6), \) and \( (7) \) allow us to conclude Theorem 1.5. For a detailed exposition of the proof we refer the reader to the original paper [6].

To study premaximality of the zero-dimensional hyperbolic sets we need the following notion. Let \( A \) be an “alphabet”: \( A = \{1, \ldots, n\} \).
and $\Omega = A^\mathbb{Z}$ equipped with the Tikhonov topology and metric
\[
d(a,b) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} I(a_i, b_i),
\]
where $a = (a_i), b = (b_i) \in \Omega$ and $I(a_i, b_i)$ equal to 0 if $a_i = b_i$ and 1 otherwise. Consider the shift map $\sigma = \Omega \to \Omega$ defined as $(\sigma(a))_i = a_{i+1}$. Consider some set $W$ of admissible words of lengths $k \geq 1$ in the alphabet $A$ and consider $M_W \subset \Omega$ such that all subwords of all $a \in M_W$ of length $k$ are admissible.

**Theorem 4.1.** Consider a set $\Lambda \subset \Omega$. The following holds

1. the set $\Lambda$ is locally maximal for $\sigma$ if and only if there exists $k \geq 1$ and set of admissible words $W$ such that $\Lambda = M_W$;
2. the set $\Lambda$ is premaximal.

In [3] it was proved that zero-dimensional hyperbolic sets are topologically conjugated to Bernoulli shifts, which implies the next result.

**Theorem 4.2.** Let $\Lambda$ be a zero-dimensional hyperbolic set of a diffeomorphism $f$. Then $\Lambda$ is premaximal.

Burns and Gelfert were able to extend the above result to prove that a 1-dimensional hyperbolic set for a flow is premaximal [8, Proposition 8]. The proof follows an argument provided by Anosov after personal communications.

In [4] Anosov obtain the following sufficient condition for a hyperbolic set to not be premaximal.

**Theorem 4.3.** Let $\Lambda$ be a hyperbolic set of $f \in C^1$. Assume that there exists a family of exact trajectories $\xi : \mathbb{Z} \times [0,a] \to M$ such that

1. $\xi_{n+1,t} = f(\xi(n,t))$,
2. $\xi(0,0) \in \Lambda$,
3. $d(\xi(n,t), \Lambda) \to 0$ uniformly in $t$ as $|n| \to \infty$, and
4. there exists $t_1 \in [0,a]$ such that $\xi(0, t_1) \notin \Lambda$.

Then $\Lambda$ is not premaximal.

Note that the examples of Crovisier and Fisher satisfy these conditions. We now prove an analog of Theorem 1.5 for flows from which Theorem 1.6 will follow.

**Theorem 4.4.** Let $\Lambda$ and $\Lambda'$ be hyperbolic sets for smooth flows $\phi$ and $\phi'$ respectively. Assume that $\Lambda$ and $\Lambda'$ are topologically equivalent, with the corresponding map $h : \Lambda \to \Lambda'$. If $U$ is a neighborhood of $\Lambda$ and $U'$ is a neighborhood of $\Lambda'$, then there exists neighborhoods $V \subset U$ of $\Lambda$
and $V' \subset U'$ of $N'$, numbers $\tau_0, \tau'_0 > 0$, and (not necessarily continuous) maps

$$h_1 : I_\phi(V) \to M' \quad \text{and} \quad h'_1 : I_{\phi'}(V') \to M$$

such that

$$h_1|_\Lambda = h \quad \text{and} \quad h'_1|_{\Lambda'} = h^{-1}. \quad (8)$$

Furthermore, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in I_\phi(V)$ and $d(x_1, x_2) < \delta$, then there exists $|\tau'| < \tau'_0$ such that

$$d(h_1(x_1), \phi_{\tau'}(h_1(x_2))) < \varepsilon. \quad (9)$$

and for any $x'_1, x'_2 \in I_{\phi'}(V')$ where $d(x'_1, x'_2) < \delta$ there exists $|\tau| < \tau_0$ such that

$$d(h'_1(x_1), \phi_{\tau}(h'_1(x_2))) < \varepsilon. \quad (10)$$

Lastly, we know

$$h_1(I_\phi(V)) \subset I_{\phi'}(U'), \quad h'_1(I_{\phi'}(V')) \subset I_\phi(U), \quad (11)$$

$$I_{\phi'}(V') \subset h_1(I_\phi(V)), \quad I_\phi(V) \subset h'_1(I_{\phi'}(V')), \quad (12)$$

and for any $x \in I_\phi(U)$ and $x' \in I_{\phi'}(U')$ we have

$$h_1(O(x)) \subset O'(h_1(x)), \quad (13)$$

$$h'_1(O'(x')) \subset O(h'_1(x')), \quad (14)$$

and there exists $|\tau| < \tau_0$ and $|\tau'| < \tau'_0$ such that

$$h'_1 \circ h_1(x) = \phi(\tau, x), \quad h_1 \circ h'_1(x') = \phi'(\tau', x'). \quad (15)$$

**Remark 4.5.** Note that (9), (10) are analogs of continuity for the maps $h_1, h'_1$. Relation (15) state that $h'_1$ is almost an inverse of $h_1$. The reason we do not obtain continuous invertible maps is due to the nonuniqueness of shadowing for flows.

**Proof.** Let $X$ be the vector field generating the smooth flow $\phi$. For a point $x \in M$ and $\varepsilon > 0$ such that $X(x) \neq 0$ denote

$$L(x, \varepsilon) := \bigcup_{v \in T_x M, |v| < \varepsilon, v \perp X(x)} \exp_x(v).$$

Note that for any closed $\phi$-invariant nonsingular set $Y$ there exists $\varepsilon, \tau_1 > 0$ and a neighborhood $O$ of $Y$ such that $O \subset \bigcup_{x \in Y} L(x, \varepsilon)$, and $L(x, \varepsilon) \cap L(\phi(t, x), \varepsilon) = \emptyset$ for $x \in M, t \in (-\tau_1, \tau_1)$.

Let $a, a', \tau_0, \tau'_0 > 0$ be constants from the expansivity property for $\Lambda$ and $\Lambda'$.
For $\delta > 0$ denote by $P(\delta)$ the set of $\delta$-pseudo orbits $g(t)$ contained in $\Lambda$. For any $\tau \in \mathbb{R}$ consider the mapping $\sigma_\tau : P(\delta) \to P(\delta)$ defined by the relation
\[
(\sigma_\tau g)(t) = g(t + \tau).
\]
Shadowing and expansivity imply that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $g \in P(\delta)$ there exists a unique point $x \in L(g(0), \varepsilon)$ and a (not necessarily unique) reparametrization $\gamma \in \text{Rep}$ such that
\[
 d(g(t), \phi(\gamma(t), x)) < \varepsilon. \tag{16}
\]
Consider $\varepsilon = a/2$ and the corresponding $\delta_0$. Now define a map $T : P(\delta_0) \to M$ such that $x = T(g)$ is the unique point satisfying (16).

For $\eta > 0$ we say that pseudo orbits $g_1, g_2 \in P(\delta)$ are $\eta$-rep-close if there exists a reparametrization $\gamma \in \text{Rep}$ such that
\[
 d(g_1(t), g_2(\gamma(t))) < \eta. \tag{17}
\]
We will use the following properties of the map $T$. By shadowing and expansivity we know that for sufficiently small $\delta$ that for any $g \in P(\delta)$ and $t \in \mathbb{R}$ there exists $t' \in \mathbb{R}$ such that
\[
 T \circ \sigma_t(g) = \phi \circ T(g) \tag{18}.
\]
Fix small enough $\eta, \delta > 0$ such that if $g_1, g_2 \in P(\delta)$ are $\eta$-rep-close then there exists $|\tau| < \tau_0$ such that
\[
 d(g_1(t), g_2(\tau, Tg_1)) < \varepsilon. \tag{19}
\]
Fix $\delta_1 > 0$ sufficiently small such that $V_0 = B(\delta_1, \Lambda) \subset U$. For any point $z \in I_\phi(V_0)$ and $t \in \mathbb{R}$ let us choose a point $g(t) \in \Lambda$ such that the inclusion $\phi_t(z) \in L(\delta_1, g(t))$ holds. We also assume that $\delta_1$ is sufficiently small so that the map $g(t)$ is a $\delta$-pseudo orbit.

Define a map $S : V_0 \to P(\delta)$ as $S(z) := g$. We would like to emphasize that the choice of $g(t)$ is not unique, however for any such choice $T(g) = z$. Again from the expansivity property we have
\[
 T \circ S = \text{id}. \tag{20}
\]
Also, for $\varepsilon > 0$ sufficiently small there exists $\Theta > 0$ such that if $g_1, g_2 \in P(\delta)$ and
\[
 d(g_1(t), g_2(t)) < \delta, \quad |t| < \Theta \tag{21}
\]
then there exists $|\tau| < \tau_0$
\[
 d(S(g_1), \phi(\tau, S(g_2))) < \varepsilon. \tag{22}
\]
For $\delta$ possibly smaller we can fix $\eta > 0$ sufficiently small so that for any $g \in P(\delta)$ pseudo orbits $g$ and $S \circ Tg$ are $\eta$-rep-close. Additionally,
there exists a neighborhood $V_1 \subset V_0$ of $\Lambda$ such that for any $x \in I_\phi(V_1)$ the inclusion $S(x) \in P(\delta)$ holds.

Similarly, fix $\delta_0', \delta_1', \delta', \eta' > 0$, maps $T' : P(\delta_0') \to M'$ and $S' : B(\delta_1', \Lambda') \to P(\delta')$, and a neighborhood $V_1'$ of $\Lambda'$.

Now we are ready to construct the maps $h_1, h_1'$. Let us choose $\delta_2 \in (0, \delta)$ such that for any $g \in P(\delta_2)$ the inclusion $h(g) \in P(\delta')$ holds. Now let us choose $V \subset V_1$ an open neighborhood of $\Lambda$ such that for any $x \in I_\phi(V)$ the inclusion $S(x) \in P(\delta_2)$ holds. Notice that if $g$ is a pseudo-orbit contained strictly in $\Lambda$, then we can define a pseudo-orbit in $\Lambda'$ by $h(g)$. We now define the map $h_1 : I_\phi(V) \to M'$ as $h_1 := T' \circ h \circ S$. Similarly, define $V'$ and map $h_1' : I_{\phi'}(V') \to M$ as $h_1' = T \circ h^{-1} \circ S'$.

Among the properties of maps $h_1, h_1'$ the most difficult is relation (15). We will give its proof in full details. Properties (13) and (14) will follow directly from the definitions of the functions $h_1$ and $h_1'$. Relations (8)–(12) can be easily deduced (decreasing $V$ and $V'$ if necessarily) from

- properties described above,
- expansivity of the vector fields in $V, V'$, and
- continuity of $h, h^{-1}$.

We prove only the second equality in (15). Note that

$$h_1' \circ h_1 = T \circ h^{-1} \circ S' \circ T' \circ h \circ S.$$

For $\eta' > 0$ perhaps smaller we know that if $g'_1$ and $g'_2$ are $\eta'$-rep-close then $h^{-1}g'_1, h^{-1}g'_2$ are $\eta$-rep-close. Also, for $\delta' > 0$ perhaps smaller we know that if $g'_1 \in P(\delta')$, then pseudo orbits $g'_1$ and $S' \circ T'g'_1$ are $\eta'$-rep-close. By the continuity of $h$ for $V$, perhaps smaller, we know that for any $x \in I_\phi(V)$ the inclusion $h \circ Sx \in P(\delta')$ holds.

Let $x \in V$. Set

$$g_1 := Sx, \quad g'_1 := hg_1, \quad x' := T'g'_1, \quad g'_2 := S'x', \quad g_2 := h^{-1}g'_2, \quad y := Tg_2.$$

By construction of $V$ we know $g'_1 \in P(\delta')$. Note that $g'_2 = S' \circ T'g'_1$. Hence, $g'_1$ and $g'_2$ are $\eta'$-rep-close. Also, we know that

$$g_1 = h^{-1}g'_1, \quad g_2 = h^{-1}g'_2.$$  

Hence, $g_1$ and $g_2$ are $\eta$-rep-close. Finally, we have

$$x = Tg_1, \quad y = Tg_2.$$

Now we know that (17) implies the second equality (15). □

**Proof of Theorem 1.6.** We prove that if $\Lambda$ is premaximal then $\Lambda'$ is premaximal. The converse statement can be proven similarly.

Consider neighborhoods $V \subset U$ of $\Lambda$ and $V' \subset U'$ of $\Lambda'$ and maps $h_1, h_1'$ from Theorem 4.4.
Assume that $\Sigma \subset V$ is a locally maximal set with isolating neighborhood $W = B(\eta, \Sigma)$, such that $\Sigma' = O(h_1(\Sigma)) \subset V'$. Below we will prove that $\Sigma'$ is locally maximal.

Consider $\eta_1 > 0$ such that for $x \in B(\eta_1, \Sigma)$ and $|\tau| < \tau_0$ the inclusion $\phi_\tau(x)$ holds. Using equality (14) and inequality (10) for $\varepsilon = \eta_1$ we find $\eta'_1 > 0$ such that if $x' \in W' := B(\eta'_1, \Sigma')$ then $h'_1(x') \in B(\eta_1, \Sigma)$.

Let us prove that $\Sigma' = I_{\phi'}W'$. Assume contrarily that there exists $x' \in W' \setminus \Sigma'$ such that $O(x') \subset W'$. Relations (13), (14), (15) imply that

$$O(h(x')) = \bigcup_{|\tau| < \tau_0, x'_1 \in O(x')} \phi_\tau(h'_1(x'_1)).$$

Note that due to choice of $\eta'_1, \eta_1$ the inclusion $\phi_\tau(h'_1(x'_1)) \in W$ holds. Hence $O(h(x')) \subset W$. Local maximality of $\Sigma$ implies that $h(x') \in \Sigma$ and $x' \in \Sigma'$, which leads to a contradiction.

Now we are ready to prove premaximality of $\Lambda'$. Since $\Lambda$ is premaximal there exists sequence $\Lambda_n \subset B(\varepsilon_n, \Lambda) \subset V$ of locally maximal sets, where $\varepsilon_n \to 0$. Arguing similarly to (18) it is easy to show that sets $\Lambda'_n = O(h_1(\Lambda_n))$ satisfies the inclusion $\Lambda'_n \subset B(\varepsilon'_n, \Lambda')$ where $\varepsilon'_n \to 0$ and for large enough $n$ sets $\Lambda'_n$ are locally maximal and contained in $U'$. Hence $\Lambda'$ is premaximal.

4.1. **Proof of Theorem 1.7** We now proceed to the proof that a hyperbolic set is premaximal if and only if the shadowing closure stabilizes. The next statement is not hard to prove, but is essential for our arguments and shows that if a sequence of shadowing closures stabilizes, then it does so at a locally maximal set.

As was claimed in [4] $\Lambda$ is locally maximal if and only if $\Lambda$ has internal shadowing (for sufficiently small $\delta > 0$ any $\delta$-pseudo orbit consisting of points from $\Lambda$ can be shadowed by a trajectory from $\Lambda$ for small enough $\delta$). It easily leads us to the following.

**Proposition 4.6.** If $\text{sh}(\Lambda, \delta) = \Lambda$ for some $\delta > 0$. Then $\Lambda$ is locally maximal.

As a reminder if $\Lambda$ is a locally maximal set, then for $\varepsilon > 0$ sufficiently small we know that $B_\varepsilon(\Lambda)$ is an isolating set for $\Lambda$.

In [9] Crovisier provides an example of a hyperbolic set that is never included in a locally maximal one, and this example shows that to establish premaximal it is not enough to say that a hyperbolic set is included in a locally maximal one or that the shadowing closure stabilizes for some $\delta > 0$. 

To be more precise, let $A$ and $B$ be $2 \times 2$ hyperbolic toral automorphisms such that $A$ has two fixed points $p$ and $q$ and the hyperbolicity in $A$ dominates the hyperbolicity in $B$. Let $F$ be a diffeomorphism on the 4-torus defined by $F_0(x, y) = (Ax, By)$ and fix $r$ a fixed point of $B$. Crovisier proves that if we let $V$ be a sufficiently small neighborhood of $(q, r)$, then the set $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(T^4 - V)$ is a hyperbolic set and the only locally maximal hyperbolic set that $\Lambda$ can be included in is the entire manifold $T^4$; so this example is not premaximal. However, $\Lambda$ is included in a locally maximal hyperbolic set and for $\delta > 0$ sufficiently large may be in a shadowing closure that stabilizes.

**Proof of Theorem 1.7.** We first consider the case of diffeomorphisms. Suppose that $\Lambda$ is a hyperbolic set for a diffeomorphism $f : M \to M$ and suppose that for any neighborhood $U$ of $\Lambda$ there exists a $\delta > 0$ such that the shadowing closure of $\Lambda$ stabilizes inside $U$. Now the previous proposition shows that the stabilizer is a locally maximal set and $\Lambda$ is premaximal.

To prove the other implication let $f : M \to M$ be a diffeomorphism and let $\Lambda$ be a premaximal set for $f$ and $V$ be a neighborhood of $\Lambda$. Then there exists some $\eta > 0$ such that $B_{\eta}(\Lambda) = \bigcup_{x \in \Lambda} B_{\eta}(x) \subset V$ and $\Lambda_{\eta} = I_f(B_{\eta}(\Lambda))$ is hyperbolic.

Let $\tilde{\Lambda}$ be a locally maximal hyperbolic set such that $\Lambda \subset \tilde{\Lambda} \subset \Lambda_{\eta} \subset B_{\eta}(\Lambda) \subset V$.

Fix $\varepsilon \in (0, \eta)$ such that $V_\varepsilon(\tilde{\Lambda})$ is an isolating neighborhood of $\tilde{\Lambda}$ and fix $\delta \in (0, \varepsilon/2)$ such that every $\delta$-pseudo orbit in $\tilde{\Lambda}$ is $\varepsilon$ shadowed in $\tilde{\Lambda}$ by a unique point in $\tilde{\Lambda}$.

From the choice of constants above we know that $\Lambda \subset \text{sh}(\Lambda, \delta) \subset \Lambda_{\varepsilon}$. Furthermore, we know that each $\delta$-pseudo orbit of $\Lambda$ is a $\delta$-pseudo orbit of $\tilde{\Lambda}$ so there exists a unique point $y \in \tilde{\Lambda}$ that is a $\varepsilon$-shadowing point of the pseudo orbit. Hence, $\text{sh}(\Lambda, \delta) \subset \tilde{\Lambda}$. Let $\Lambda_1 = \text{sh}(\Lambda, \delta)$.

If $\Lambda_1 = \Lambda$ we know from the previous proposition that $\Lambda$ is locally maximal. So suppose that $\Lambda_1 \neq \Lambda$ and let $\nu_1 = d_H(\Lambda_1, \Lambda)$ where $d_H(\cdot, \cdot)$ is the Hausdorff distance between the sets. More generally, let $\Lambda_{j+1} = \text{sh}(\Lambda_j, \delta)$, $\nu_{j+1} = d_H(\Lambda_{j+1}, \Lambda_j)$ for $j \in \mathbb{N}$. By the shadowing estimates we know that $\nu_j \in (0, \varepsilon)$. 

Claim 4.7. There exists $\gamma > 0$ such that for all $j \in \mathbb{N}$ if $\Lambda_j \neq \Lambda_{j+1}$ and $\Lambda_{j+1} \neq \Lambda_{j+2}$, then either $\nu_j \geq \gamma$ or $\nu_{j+1} \geq \gamma$.

Proof of Claim. Consider $\gamma \in (0, \delta/4)$ such that for all $x, y \in M$ if $d(x, y) < \gamma$, then

$$d(f(x), f(y)) < \delta/4 \text{ and } d(f^{-1}(x), f^{-1}(y)) < \delta/4.$$ 

Suppose that $\Lambda_j \neq \Lambda_{j+1}$ and $\Lambda_{j+1} \neq \Lambda_{j+2}$ and $\nu_j, \nu_{j+1} \in (0, \gamma)$. Fix $y \in \Lambda_{j+2} - \Lambda_{j+1}$. We will construct a $\delta$-pseudo orbit in $\Lambda_j$ that $y$ $\varepsilon$-shadows. This will show that $y \in \Lambda_{j+1}$, a contradiction.

Then there exists some $y_0 \in \Lambda_{j+1}$ and $x_0 \in \Lambda_j$ such that $d(y_0, x_0) < \nu_{j+1}$ and $d(y_0, x_0) < \nu_j < \gamma$. Then

$$d(f(y), f(y_0)) < \delta/4 \text{ and } d(f(y_0), f(x_0)) < \delta/4.$$ 

Also, we know there exists points $y_1 \in \Lambda_{j+1}$ and $x_1 \in \Lambda_j$ such that $d(f(y), y_1) < \nu_{j+2}$ and $d(y_1, x_1) < \nu_{j+1}$. Hence,

$$d(f(x_0), x_1) \leq d(f(x_0), f(y_0)) + d(f(y_0), f(y)) + d(f(y), y_1) + d(y_1, x_1) \leq \delta/4 + \delta/4 + \nu_{j+2} + \nu_{j+1} < \delta$$

and

$$d(f(y), x_1) \leq d(f(y), y_1) + d(y_1, x_1) < \nu_{j+2} + \nu_{j+1} < \delta/2 < \varepsilon.$$ 

Continue inductively, we can construct a forward $\delta$-pseudo orbit $(x_k)_{k=0}^{\infty}$ such that $y$ $\varepsilon$-shadows the pseudo orbit. Also, since the estimates on $\gamma$ apply for $f^{-1}$ we can construct a bi-infinite $\delta$-pseudo orbit $(x_k)$ in $\Lambda_j$ such that $y$ $\varepsilon$ shadows $(x_k)$. Then $y \in \Lambda_{j+1}$, a contradiction. \hfill \Box

We now return to the proof of the theorem. Repeating the above arguments for $\Lambda_j$ we see that $\Lambda_j \subset \tilde{\Lambda}$ for all $j \in \mathbb{N}$. We know from Proposition 1.6 that if $\Lambda_{j+1} = \Lambda_j$ for some $j \in \mathbb{N}$ that $\Lambda_j$ is locally maximal. To conclude the proof of the theorem we simply need to show that the sequence $\Lambda_j$ stabilizes.

Suppose that the sequence $\Lambda_j$ does not stabilize. We know from the above claim that for each $j$ the sets $\Lambda_{j+1}$ or $\Lambda_{j+2}$ will be a distance of $\gamma$ from the set $\Lambda_j$ using the Hausdorff metric. Inside a compact metric space we know an increasing sequence of compact sets converges in the Hausdorff topology. Hence, if the sequence does not stabilize there exists some $N \in \mathbb{N}$ where the Hausdorff distance from $\Lambda_N$ to $\Lambda$ is greater than $\eta$. This is a contradiction since each $\Lambda_j \subset \tilde{\Lambda}$ and $\tilde{\Lambda} \subset V_\eta(\Lambda)$. Theorem 1.7 is proved for the case of diffeomorphisms.

The proof for the case of flows follows the same ideas. Below we provide a detailed proof of the Claim 4.7 which is the central step of the proof of Theorem 1.7.
We will use the following construction. For a sequence of points \( \{x_k\}_{k \in \mathbb{Z}} \) consider a map \( g_{\{x_k\}} : \mathbb{R} \to M \) defined as
\[
g_{\{x_k\}}(t) := \phi(t - [t], x_{[t]}),
\]
where \([t]\) is the integer part of \( t \).

Consider map \( r : M \to M \) defined as \( r(x) = \phi(1, x) \). Note that \( r \) does not necessarily satisfy the shadowing property. We will use the following.

**F1:** There exists \( \varepsilon_1 > 0 \) such that for any sequence \( \{x_k\} \) and point \( y \) satisfying
\[
d(x_k, f^k(y)) < \varepsilon_1, \quad k \in \mathbb{Z}
\]
the following inequality holds
\[
d(g_{\{x_k\}}(t), \phi(t, y)) < \varepsilon.
\]
Without loss of generality we can assume that \( \delta < \varepsilon_1 \).

**F2:** There exists \( \delta_1 > 0 \) such that if \( \{x_k\} \) with \( x_k \in \Lambda_j \) is a \( \delta_1 \)-pseudo orbit of the map \( r \) then \( g_{\{x_k\}} \subset \Lambda_j \) is a \( \delta \)-pseudo orbit of the flow \( \phi \).

Consider \( \gamma \in (0, \delta_1/4) \) such that for all \( x, y \in M \) if \( d(x, y) < \gamma \), then
\[
d(r(x), r(y)) < \delta_1/4 \text{ and } d(r^{-1}(x), r^{-1}(y)) < \delta_1/4.
\]
As in the case of diffeomorphisms suppose that \( \Lambda_j \neq \Lambda_{j+1} \) and \( \Lambda_{j+1} \neq \Lambda_{j+2} \) and \( \nu_j, \nu_{j+1} \in (0, \gamma) \). Fix \( y \in \Lambda_{j+2} - \Lambda_{j+1} \).

Arguing similarly to the case of diffeomorphisms we can construct a \( \delta_1 \)-pseudo orbit \( \{x_k\} \subset \Lambda_j \) of the map \( r \) which satisfies \([19]\). Properties **F1** and **F2** implies that \( \delta \)-pseudo orbit \( g_{\{x_k\}} \) is \( \varepsilon \)-shadowed by \( y \). This shows that \( y \in \Lambda_{j+1} \), a contradiction.

\[\square\]

**References**


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