Hyperbolic Dynamics

Todd Fisher

tfisher@math.umd.edu

Department of Mathematics
University of Maryland, College Park
What is a dynamical system?

- Phase space $X$, elements possible states
What is a dynamical system?

- Phase space $X$, elements possible states
- Time, discrete or continuous ($\mathbb{Z}$ or $\mathbb{R}$)
What is a dynamical system?

- Phase space $X$, elements possible states
- Time, discrete or continuous ($\mathbb{Z}$ or $\mathbb{R}$)
- Action on the space
Class we will look at

Today:

- time discrete (e.g. - time-t map of a flow),
Class we will look at

Today:
- time discrete (e.g., time-t map of a flow),
- \( X \) a manifold (think of \( \mathbb{R}^n \) or sphere, circle, torus.)
Class we will look at

Today:

- time discrete (e.g.- time-t map of a flow),
- $X$ a manifold (think of $\mathbb{R}^n$ or sphere, circle, torus.)
- $f : X \rightarrow X$ a smooth map.
Class we will look at

Today:

- time discrete (e.g. - time-t map of a flow),
- $X$ a manifold (think of $\mathbb{R}^n$ or sphere, circle, torus.)
- $f : X \to X$ a smooth map.

The pair $(X, f)$ is a dynamical system. Study action of $f$ on $X$ under iteration.
Class we will look at

Today:

- time discrete (e.g.- time-t map of a flow),
- $X$ a manifold (think of $\mathbb{R}^n$ or sphere, circle, torus.)
- $f : X \to X$ a smooth map.

The pair $(X, f)$ is a dynamical system. Study action of $f$ on $X$ under iteration.

**Advantage of smooth:** derivative gives idea of local behavior.
What is the study of dynamical systems?

In dynamical systems we look at asymptotic behavior of a system.

For instance:

- Many applications - disease, populations, ocean currents
What is the study of dynamical systems?

In dynamical systems we look at asymptotic behavior of a system.

For instance:

- Many applications - disease, populations, ocean currents
- ODE - do solutions go to equilibrium? If not what do they do?
Orbit structure

For orbit of a point we iterate.

\[ x, f(x), f^2(x) = f(f(x)), \ldots, f^j(x), \ldots \]

- Interested in global orbit structure.
For orbit of a point we iterate.

\[ x, f(x), f^2(x) = f(f(x)), ..., f^j(x), ... \]

- Interested in global orbit structure.
- Properties of orbit (periodicity, recurrence, non-recurrence).
Orbit structure

For orbit of a point we iterate.

\[ x, f(x), f^2(x) = f(f(x)), \ldots, f^j(x), \ldots \]

- Interested in global orbit structure.
- Properties of orbit (periodicity, recurrence, non-recurrence).
- “Typical” behavior of orbits (typical can be dense, open and dense, almost everywhere).
Orbit structure

For orbit of a point we iterate.

\[ x, f(x), f^2(x) = f(f(x)), \ldots, f^j(x), \ldots \]

- Interested in global orbit structure.
- Properties of orbit (periodicity, recurrence, non-recurrence).
- “Typical” behavior of orbits (typical can be dense, open and dense, almost everywhere).
- Statistical properties of orbits.
1. \( f : S^1 \to S^1 \) given by

\[
f(x) = (x + \alpha) \mod(1)
\]
Examples

1. $f : S^1 \to S^1$ given by

$$f(x) = (x + \alpha) \mod(1)$$

$\alpha$ rational all points periodic. $\alpha$ irrational all points have dense orbit
Examples

1. $f : S^1 \to S^1$ given by

$$f(x) = (x + \alpha) \text{mod}(1)$$

$\alpha$ rational all points periodic. $\alpha$ irrational all points have dense orbit

2.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

This matrix has determinant 1 so induces diffeomorphism $f_A$ of $T^2 \cong \mathbb{R}^2 / \mathbb{Z}^2$. 
Definition: Two maps $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are \textit{topologically conjugate} if there is a homeomorphism $h : X \rightarrow Y$ such that $hf = gh$. 
Definition: Two maps $f: X \rightarrow X$ and $g: Y \rightarrow Y$ are topologically conjugate if there is a homeomorphism $h: X \rightarrow Y$ such that $hf = gh$.

Remark: This is common notion of equivalence for dynamical systems. Requiring $h$ to be a diffeomorphism is too restrictive.
Hyperbolic fixed point

**Definition:** A fixed point is *hyperbolic* if none of the eigenvalues of the derivative lie on unit circle.
Hyperbolic fixed point

**Definition:** A fixed point is *hyperbolic* if none of the eigenvalues of the derivative lie on unit circle.

**Note:** This is a robust condition.
Definition: A fixed point is *hyperbolic* if none of the eigenvalues of the derivative lie on unit circle.

Note: This is a robust condition.

Hartman-Grobman Theorem says if $\dim(M) = n$, $f \in \text{Diff}(M)$, and $p$ is a hyperbolic fixed point of $f$, then there exists a homeomorphism from a neighborhood of $p$ to neighborhood of the origin in $\mathbb{R}^n$ linearizing the dynamics to the derivative of $f$ at $p$. 
A compact set $\Lambda$ is *hyperbolic* if it is invariant ($f(\Lambda) = \Lambda$) and the tangent space has a continuous invariant splitting $T_{\Lambda} M = E^s \oplus E^u$ where $E^s$ is uniformly contracting and $E^u$ is uniformly expanding under the derivative.
A compact set $\Lambda$ is \textit{hyperbolic} if it is invariant ($f(\Lambda) = \Lambda$) and the tangent space has a continuous invariant splitting $T_{\Lambda}M = E^s \oplus E^u$ where $E^s$ is uniformly contracting and $E^u$ is uniformly expanding under the derivative.

Compact + Hyperbolic = Interesting Dynamics
A compact set $\Lambda$ is hyperbolic if it is invariant ($f(\Lambda) = \Lambda$) and the tangent space has a continuous invariant splitting $T_\Lambda M = E^s \oplus E^u$ where $E^s$ is uniformly contracting and $E^u$ is uniformly expanding under the derivative.

Compact + Hyperbolic = Interesting Dynamics

Provides good start to global structure.
A compact set $\Lambda$ is *hyperbolic* if it is invariant ($f(\Lambda) = \Lambda$) and the tangent space has a continuous invariant splitting $T_{\Lambda}M = E^s \oplus E^u$ where $E^s$ is uniformly contracting and $E^u$ is uniformly expanding under the derivative.

Compact + Hyperbolic = Interesting Dynamics

Provides good start to global structure.

Derivative gives good local description of the map.
Structure of Hyperbolic Sets

**Question:** If $\Lambda$ is a hyperbolic set what is the topology of $\Lambda$? what is the dynamics of $f$ on $\Lambda$? what is the dynamics of $f$ in a neighborhood of $\Lambda$?
Structure of Hyperbolic Sets

**Question:** If $\Lambda$ is a hyperbolic set what is the topology of $\Lambda$? What is the dynamics of $f$ on $\Lambda$? What is the dynamics of $f$ in a neighborhood of $\Lambda$?

Two-pronged approach:
Structure of Hyperbolic Sets

**Question:** If $\Lambda$ is a hyperbolic set what is the topology of $\Lambda$? what is the dynamics of $f$ on $\Lambda$? what is the dynamics of $f$ in a neighborhood of $\Lambda$?

**Two-pronged approach:**

- What is the topology of $\Lambda$
- Assuming some knowledge of the topology, what are the dynamics?
Different examples with classification

- Locally maximal hyperbolic sets
Different examples with classification

- Locally maximal hyperbolic sets
- Hyperbolic attractors
Different examples with classification

- Locally maximal hyperbolic sets
- Hyperbolic attractors
- Submanifolds that are hyperbolic sets
Different examples with classification

- Locally maximal hyperbolic sets
- Hyperbolic attractors
- Submanifolds that are hyperbolic sets
- Hyperbolic sets with non-empty interior
What about when hyperbolicity fails

I will show that hyperbolicity arises in many applications. However, hyperbolicity of the recurrent set is not dense. So an additional question will be the following:
What about when hyperbolicity fails

I will show that hyperbolicity arises in many applications. However, hyperbolicity of the recurrent set is not dense. So an additional question will be the following:

**Question:** If the recurrent set is not hyperbolic can we describe the typical dynamics?
Stable and unstable sets

The study of the structure of hyperbolic sets is intrinsically related with the structure of the points that under forward or backward iterates converge on the set.

Definition:
The stable set of a point $x$ is $W_s(x) = \{ y \in X : f^n(y) \to x \}$ as $n \to \infty$.

Definition:
The unstable set of a point $x$ is $W_u(x) = \{ y \in X : f^{-n}(y) \to x \}$ as $n \to \infty$.
The study of the structure of hyperbolic sets is intrinsically related with the structure of the points that under forward or backward iterates converge on the set.

**Definition:** The *stable set* of a point \( x \) is

\[
W^s(x) = \{ y \in M \mid d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty \}.
\]

**Definition:** The *unstable set* of a point \( x \) is

\[
W^u(x) = \{ y \in M \mid d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty \}.
\]
Outline of the rest of the talk

- Origins - Poincare and Smale
Outline of the rest of the talk

- Origins - Poincaré and Smale
- Properties - structural stability and shadowing
Outline of the rest of the talk

- Origins - Poincare and Smale
- Properties - structural stability and shadowing
- Classification results
Origins

Origins


- Poincare (1890) found hyperbolic phenomenon, unprecedented complexity (reversed thrust of work).
Let $p$ be hyperbolic fixed point and $x \in W^s(p) \cap W^u(p)$ is called a transverse homoclinic point.
Homiclinic point

Let $p$ be a hyperbolic fixed point and $x \in W^s(p) \cap W^u(p)$ is called a transverse homoclinic point.
Let $p$ be a hyperbolic fixed point and $x \in W^s(p) \cap W^u(p)$ is called a transverse homoclinic point.
Let $p$ be hyperbolic fixed point and $x \in W^s(p) \cap W^u(p)$ is called a transverse homoclinic point.
Properties of homoclinic point

Poincare and others noticed properties of homoclinic point. Since intersection is transverse we know it is robust.

Let $U$ be neighborhood of $\mathcal{O}(x) \cup p$.

∃ hyperbolic set $\Lambda \subset U$ such that:

- periodic points dense in $\Lambda$
Poincare and others noticed properties of homoclinic point. Since intersection is transverse we know it is robust.

Let $U$ be neighborhood of $\mathcal{O}(x) \cup p$.

$\exists$ hyperbolic set $\Lambda \subset U$ such that:

- periodic points dense in $\Lambda$
- $\Lambda$ has point with dense orbit.
Poincare and others noticed properties of homoclinic point. Since intersection is transverse we know it is robust.

Let $U$ be neighborhood of $\mathcal{O}(x) \cup p$.

$\exists$ hyperbolic set $\Lambda \subset U$ such that:

- periodic points dense in $\Lambda$
- $\Lambda$ has point with dense orbit.
- $\Lambda$ has positive entropy. (Value measures complexity.)
Smale’s work in the 60’s

Definition: A diffeomorphism $f$ is *structurally stable* if there exists a neighborhood $U$ such that for each $g \in U$ is topologically conjugate to $f$.

- Smale began classification of diffeomorphisms.
Smale’s work in the 60’s

**Definition:** A diffeomorphism $f$ is *structurally stable* if there exists a neighborhood $U$ such that for each $g \in U$ is topologically conjugate to $f$.

- Smale began classification of diffeomorphisms.
- Conjectured structurally stable were dense.
Smale’s work in the 60’s

**Definition:** A diffeomorphism $f$ is *structurally stable* if

$\exists$ neighborhood $U$ such that for each $g \in U$ is topologically conjugate to $f$.

- Smale began classification of diffeomorphisms.
- Conjectured structurally stable were dense.
- Thought all structurally stable were dynamically trivial (Morse-Smale - like time-t map of gradient flow).
Work on conjecture

Levinson told Smale about Poincare’s work.
Levinson told Smale about Poincare’s work. Smale generalized Poincare’s work, defined hyperbolic set. Follows that hyperbolic sets are structurally stable.

Theorem:
If $f : \mathbb{R}^{n} \to \mathbb{R}^{n}$ and $\mathcal{O}$ is hyperbolic for $f$, then there exists a neighborhood $U$ of $f$ in $\text{Diff}^{\infty}(\mathbb{R}^{n})$ such that for each $g \in U$ there exists a homeomorphism $h : \mathcal{O} \to \mathcal{O}(g)$ such that $\mathcal{O}(g)$ is a hyperbolic set for $g$.

Note: $\mathcal{O}(g)$ is called the continuation of $\mathcal{O}$ for $g$. 
Levinson told Smale about Poincare’s work. Smale generalized Poincare’s work, defined hyperbolic set. Follows that hyperbolic sets are structurally stable.

**Theorem:** If \( f \in \text{Diff}(M) \) and \( \Lambda \subset M \) is hyperbolic for \( f \), then \( \exists \) neighborhood \( U \) of \( f \) in \( \text{Diff}(M) \) such that for each \( g \in U \) \( \exists \) homeomorphism \( h(g) : \Lambda \rightarrow \Lambda(g) \) such that \( \Lambda(g) \) is a hyperbolic set for \( g \).

**Note:** \( \Lambda(g) \) is called the continuation of \( \Lambda \) for \( g \).
Weak Palis Conjecture

**Note:** Recent Theorem says transverse homoclinic points are very common for diffeomorphisms.

**Theorem 1** *(Weak Palis Conjecture)* For any smooth manifold there is an open and dense set of $C^1$ diffeomorphisms that are either Morse-Smale (dynamically trivial) or contain a transverse homoclinic point.

Proof announced by Crovisier, based on work of Bonatti, Gan, and Wen.
A hyperbolic set $\Lambda$ is **locally maximal** (or isolated) if there exists an open set $U$ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

These have many useful properties help to understand structure.
A hyperbolic set \( \Lambda \) is **locally maximal** (or isolated) if there exists an open set \( U \) such that

\[
\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).
\]

These have many useful properties help to understand structure.

**Question 1:** (Katok) “Let \( \Lambda \) be a hyperbolic set... and \( V \) an open neighborhood of \( \Lambda \). Does there exist a locally maximal hyperbolic set \( \tilde{\Lambda} \) such that \( \Lambda \subset \tilde{\Lambda} \subset V \) ?”
Answer to question

Crovisier(2001) answers no for specific example on four torus.

Theorem

(F.) On any compact manifold $M$, where $\dim(M) \geq 2$, there exists a $C^1$ open set of diffeomorphisms, $U$, such that any $f \in U$ has a hyperbolic set that is not contained in a locally maximal hyperbolic set. Furthermore, in 4-dimensions hyperbolic set can have point with dense orbit.
Answer to question

Crovisier(2001) answers no for specific example on four torus.

**Theorem** (F.) On any compact manifold $M$, where $\dim(M) \geq 2$, there exists a $C^1$ open set of diffeomorphisms, $U$, such that any $f \in U$ has a hyperbolic set that is not contained in a locally maximal hyperbolic set.

Furthermore, in 4-dimensions hyperbolic set can have point with dense orbit.
Answer to question

Crovisier(2001) answers no for specific example on four torus.

**Theorem (F.)** On any compact manifold $M$, where $\dim(M) \geq 2$, there exists a $C^1$ open set of diffeomorphisms, $U$, such that any $f \in U$ has a hyperbolic set that is not contained in a locally maximal hyperbolic set.

Furthermore, in 4-dimensions hyperbolic set can have point with dense orbit.
Locally maximal hyperbolic sets have Markov partition.
Markov Partition

Locally maximal hyperbolic sets have Markov partition.

A *Markov partition* is a decomposition of $\Lambda$ into dynamically defined rectangles.
Markov Partition

Locally maximal hyperbolic sets have Markov partition.

A *Markov partition* is a decomposition of $\Lambda$ into dynamically defined rectangles.

Associated with partition is symbolic dynamical system. Properties of symbolic system carry over to $\Lambda$. 
Markov Partition

Locally maximal hyperbolic sets have Markov partition.

A *Markov partition* is a decomposition of $\Lambda$ into dynamically defined rectangles.

Associated with partition is symbolic dynamical system. Properties of symbolic system carry over to $\Lambda$.

**Theorem:** (F.) If $\Lambda$ is a hyperbolic set and $V$ is a neighborhood of $\Lambda$, then there exists a hyperbolic set $\tilde{\Lambda}$ with a Markov partition such that $\Lambda \subset \tilde{\Lambda} \subset V$. 
Proof of the existence of Markov partitions uses shadowing.

Shadowing Diagram
Proof of the existence of Markov partitions uses shadowing.
Proof of the existence of Markov partitions uses shadowing.
Proof of the existence of Markov partitions uses shadowing.
Proof of the existence of Markov partitions uses shadowing.
Proof of the existence of Markov partitions uses shadowing.
Shadowing for hyperbolic sets

**Theorem:** If $\Lambda$ is a hyperbolic set and $\delta > 0$, there exists an $\varepsilon > 0$ such that if $\{x_i\}$ for $-\infty \leq j_1 \leq i \leq j_2 \leq \infty$ is an $\varepsilon$-pseudo orbit, then there exists a point $y$ that $\delta$-shadows $\{x_i\}$. Furthermore, if $j_1 = -1$ and $j_2 = 1$, then $y$ is unique.

Note: If $\Lambda$ is locally maximal, $y$ can be shadowed by a point in $\Lambda$. 
Shadowing for hyperbolic sets

**Theorem:** If $\Lambda$ is a hyperbolic set and $\delta > 0$, there exists an $\epsilon > 0$ such that if $\{x_i\}$ for $-\infty \leq j_1 \leq i \leq j_2 \leq \infty$ is an $\epsilon$-pseudo orbit, then there exists a point $y$ that $\delta$-shadows $\{x_i\}$.

Furthermore, if $j_1 = -\infty$ and $j_2 = \infty$, then $y$ is unique.
Theorem: If $\Lambda$ is a hyperbolic set and $\delta > 0$, there exists an $\epsilon > 0$ such that if $\{x_i\}$ for $-\infty \leq j_1 \leq i \leq j_2 \leq \infty$ is an $\epsilon$-pseudo orbit, then there exists a point $y$ that $\delta$-shadows $\{x_i\}$.

Furthermore, if $j_1 = -\infty$ and $j_2 = \infty$, then $y$ is unique.

Note: If $\Lambda$ is locally maximal $y$ can be shadowed by a point in $\Lambda$. 
Another important class of hyperbolic sets are attractors. Attractors arise in many applications. Structure still only understood in certain cases.
Another important class of hyperbolic sets are attractors. Attractors arise in many applications. Structure still only understood in certain cases.

Definition: A neighborhood $U$ of a compact set $X$ is an attracting set if

$$X = \bigcap_{n \in \mathbb{N}} f^n(U).$$
Definition: A hyperbolic attractor is a hyperbolic set with a dense orbit and an attracting set.
Hyperbolic attractors

**Definition:** A hyperbolic attractor is a hyperbolic set with a dense orbit and an attracting set.

**Conjecture:** (Palis) For a typical diffeomorphism there is a finite number of attractors such that an open and dense set of points are contained in the union of the basins for the attractors.

Note: Hyperbolic attractors with one-dimensional stable splitting have been characterized. (Williams, Plykin, Grines and Zhuzhoma).

Remark: Hyperbolic attractors known to have nice statistical properties (SRB measures).
Definition: A *hyperbolic attractor* is a hyperbolic set with a dense orbit and an attracting set.

Conjecture: (Palis) For a typical diffeomorphism there is a finite number of attractors such that an open and dense set of points are contained in the union of the basins for the attractors.

Note: Hyperbolic attractors with one-dimensional stable splitting have been characterized. (Williams, Plykin, Grines and Zhuzhoma).
Hyperbolic attractors

Definition: A hyperbolic attractor is a hyperbolic set with a dense orbit and an attracting set.

Conjecture: (Palis) For a typical diffeomorphism there is a finite number of attractors such that an open and dense set of points are contained in the union of the basins for the attractors.

Note: Hyperbolic attractors with one-dimensional stable splitting have been characterized. (Williams, Plykin, Grines and Zhuzhoma).

Remark: Hyperbolic attractors known to have nice statistical properties (SRB measures).
Hyperbolic Attractors on Surfaces

Theorem: (F.) If $M$ is a compact smooth surface, $\Lambda$ is a mixing hyperbolic attractor for $f$, and $\Lambda$ is a hyperbolic for $g$, then $\Lambda$ is either a hyperbolic attractor or repeller for $g$. 

Note: Mixing is a topological property stronger than having a point with a dense orbit. If we know the set and we know that it is hyperbolic we know it is an attractor.
Theorem: (F.) If $M$ is a compact smooth surface, $\Lambda$ is a mixing hyperbolic attractor for $f$, and $\Lambda$ is a hyperbolic for $g$, then $\Lambda$ is either a hyperbolic attractor or repeller for $g$.

Note: Mixing is topological property stronger than having a point with a dense orbit.
Theorem: (F.) If $M$ is a compact smooth surface, $\Lambda$ is a mixing hyperbolic attractor for $f$, and $\Lambda$ is a hyperbolic for $g$, then $\Lambda$ is either a hyperbolic attractor or repeller for $g$.

Note: Mixing is topological property stronger than having a point with a dense orbit.

If we know the the set and we know that it is hyperbolic we know it is an attractor.
Anosov diffeomorphisms

Definition: A diffeomorphism $f \in \text{Diff}(M)$ is Anosov if $M$ is a hyperbolic set for $f$. 

Note: In the Anosov case $M$ is trivially an attractor if $f$ is transitive.

Result of Franks and Manning says that any Anosov diffeomorphism on an infranilmanifold is topologically conjugate to hyperbolic infranilmanifold automorphism.

Not known if all Anosov diffeomorphisms on infranilmanifolds.
Anosov \textbf{diffeomorphisms}

\textbf{Definition:} A diffeomorphism $f \in \text{Diff}(M)$ is \textit{Anosov} if $M$ is a hyperbolic set for $f$.

\textbf{Note:} In the Anosov case $M$ is trivially an attractor if $f$ is transitive.
Definition: A diffeomorphism \( f \in \text{Diff}(M) \) is \textit{Anosov} if \( M \) is a hyperbolic set for \( f \).

Note: In the Anosov case \( M \) is trivially an attractor if \( f \) is transitive.

Result of Franks and Manning says that any Anosov diffeomorphism on a infranilmanifold is topologically conjugate to hyperbolic infranilmanifold automorphism.
**Anosov diffeomorphisms**

**Definition:** A diffeomorphism \( f \in \text{Diff}(M) \) is *Anosov* if \( M \) is a hyperbolic set for \( f \).

**Note:** In the Anosov case \( M \) is trivially an attractor if \( f \) is transitive.

Result of Franks and Manning says that any Anosov diffeomorphism on a infranilmanifold is topologically conjugate to hyperbolic infranilmanifold automorphism.

Not known if all Anosov diffeomorphisms on infranilmanifolds.
Hyperbolic submanifolds

Question: (Hirsch, ’69) If $f \in \text{Diff}(N)$ and $M \subset N$ a submanifold is a hyperbolic set, then what do we know about $M$? About $f|_M$? 

If $\dim(M) = 2$, then $M$ is $T^2$ and $f|_M$ is Anosov. (Mañé).

If $\dim(M) = 3$, Franks and Robinson (’76) show example not Anosov.
Question: (Hirsch, ’69) If $f \in \text{Diff}(N)$ and $M \subset N$ a submanifold is a hyperbolic set, then what do we know about $M$? About $f|_M$?

If $\dim(M) = 2$, then $M$ is $\mathbb{T}^2$ and $f|_M$ is Anosov. (Mañé).
Question: (Hirsch, ’69) If $f \in \text{Diff}(N)$ and $M \subset N$ a submanifold is a hyperbolic set, then what do we know about $M$? About $f|_M$?

If $\dim(M) = 2$, then $M$ is $\mathbb{T}^2$ and $f|_M$ is Anosov. (Mañé).

If $\dim(M) = 3$, Franks and Robinson (’76) show example not Anosov.
3-manifolds

Theorem: (F., J. Rodriguez-Hertz) If $f \in \text{Diff}(N)$, $M \subset N$, $\dim(M) = 3$, and hyperbolic for $f$, then $\Lambda$ is the connected sum of copies of $T^3$. 
3-manifolds

**Theorem:** (F., J. Rodriguez-Hertz) If $f \in \text{Diff}(N)$, $M \subset N$, $\dim(M) = 3$, and hyperbolic for $f$, then $\Lambda$ is the connected sum of copies of $T^3$.

Furthermore, if $f|_M$ is not Anosov, then the dynamics on each copy of $M$ is conjugate to DA-diffeomorphism on punctured torus.
Since hyperbolic sets have strong expansion and contraction. Then non-empty interior should give strong conditions on $\Lambda$.

**Question:** If $\Lambda$ has non-empty interior does $\Lambda = M$?
Since hyperbolic sets have strong expansion and contraction. Then non-empty interior should give strong conditions on $\Lambda$.

**Question:** If $\Lambda$ has non-empty interior does $\Lambda = M$?

This would give sufficient condition for a system to be Anosov that is verifiable at just one point.
Hyperbolic Sets with Interior

Since hyperbolic sets have strong expansion and contraction. Then non-empty interior should give strong conditions on $\Lambda$.

**Question:** If $\Lambda$ has non-empty interior does $\Lambda = M$?

This would give sufficient condition for a system to be Anosov that is verifiable at just one point.

**Theorem:** (F.) There exist hyperbolic sets with non-empty interior that are not Anosov.
What conditions imply Anosov?

**Theorem:** (F.) If $\Lambda$ is a hyperbolic set with nonempty interior, then $f$ is Anosov ($\Lambda = M$) if

1. $\Lambda$ is transitive
2. $\Lambda$ is locally maximal and $M$ is a surface
What conditions imply Anosov?

Theorem: (F.) If $\Lambda$ is a hyperbolic set with nonempty interior, then $f$ is Anosov ($\Lambda = M$) if

1. $\Lambda$ is transitive
2. $\Lambda$ is locally maximal and $M$ is a surface

Removing surface in second part would solve long standing open question.
Lack of hyperbolicity

**Definition:** The *homoclinic class* of a hyperbolic periodic point is the closure of the set of transverse homoclinic points.
Definition: The *homoclinic class* of a hyperbolic periodic point is the closure of the set of transverse homoclinic points.

Hyperbolic homoclinic classes are well understood.
Lack of hyperbolicity

**Definition:** The *homoclinic class* of a hyperbolic periodic point is the closure of the set of transverse homoclinic points.

Hyperbolic homoclinic classes are well understood.

Since hyperbolicity of recurrent set is not dense among diffeomorphisms. Then we are also interested in non-hyperbolic homoclinic classes.
Non-hyperbolic homoclinic class

**Theorem:** (Bonatti, Diaz, F.) There is a dense subset (residual) $\mathcal{R}$ of $\text{Diff}^1(M)$ such that for every $f \in \mathcal{R}$ any homoclinic class of $f$ containing hyperbolic saddles with different number of stable directions has superexponential growth of the number of periodic points.
Theorem: (Bonatti, Diaz, F.) There is a dense subset (residual) $\mathcal{R}$ of $\text{Diff}^1(M)$ such that for every $f \in \mathcal{R}$ any homoclinic class of $f$ containing hyperbolic saddles with different number of stable directions has superexponential growth of the number of periodic points.

For hyperbolic homoclinic classes only exponential growth.
References

For preprints and copy of presentation
www.math.umd.edu/~tfisher