Quasi-Anosov diffeomorphisms of 3-manifolds

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1 Motivation

2 Example of non-Anosov QAD in 3-dimensions

3 Results
Basic question

Question

(Hirsch, ’71) If $N$ is a closed smooth manifold, $f \in \text{Diff}^1(N)$, and $M$ is a closed smooth submanifold of $N$ such that $M$ is a hyperbolic set for $f$, then is $f|_M$ Anosov?
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We will see the answer is no (Franks, Robinson 76).
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Problem

Let $M$ be a closed smooth submanifold of $N$, where $\dim(M) = 3$, that is a hyperbolic set for some diffeomorphism $f$ of $N$. We want to classify geometry of $M$ and dynamics of $M$ under the action of $f$.

Remark

This is joint work with J. Rodriguez-Hertz.
Quasi-Anosov diffeomorphisms

Definition

A diffeomorphism of a closed smooth manifold $M$ is quasi-Anosov (QAD) if for all $0 \neq v \in TM$ the set $\{ \| (Tf)^n v \| \mid n \in \mathbb{Z} \}$ is unbounded.
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(Mañé, ’77) The following are equivalent:

1. \( f \) is QAD.
2. \( f \) is Axiom A with no cycles and for all \( x \in M \) there is a quasi-transverse intersection for the stable and unstable manifolds \( (T_x W^s(x) \cap T_x W^u(x) = \{0\}) \).
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3. There exists a smooth manifold $N$ and smooth embedding $i : M \rightarrow N$ and $g \in \text{Diff}^r(N)$ such that $g \circ i = i \circ f$ and $i(M)$ is a hyperbolic set for $g$. 
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3. There exists a smooth manifold $N$ and smooth embedding $i : M \to N$ and $g \in \text{Diff}^r(N)$ such that $g \circ i = i \circ f$ and $i(M)$ is a hyperbolic set for $g$.
4. $f$ is robustly expansive.
Definition

A diffeomorphism is expansive if there exists some $\alpha > 0$ such that \( \sup_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) \leq \alpha \) implies $x = y$.

One reason to study QAD’s and the manifolds that support QAD’s is the hope that this will help one see restrictions expansive diffeomorphisms impose on manifolds.
Restatement of question

From above Hirsch’s question is:

**Question**

*Does QAD imply Anosov?*
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**Question**
*Does QAD imply Anosov?*

Franks and Robinson (’76) provide 3-dimensional example of QAD that is not Anosov.

**Remark**
*In 2-dimension all QAD are Anosov. Since QAD are Axiom A with quasi-transverse intersections. So we have splittings that are all 1 − 1 and no where tangent.*
When is a QAD also Anosov?

Corollary

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**Definition**

A hyperbolic set \( \Lambda \) is locally maximal (or isolated) if there exists a neighborhood \( U \) of \( \Lambda \) such that \( \bigcap_{n \in \mathbb{Z}} f^n(U) = \Lambda \).
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Definition

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Corollary

If $M$ is a hyperbolic manifold for $f \in \text{Diff}^1(N)$, $f|_M$ is Anosov if and only if $M$ is locally maximal.
Outline

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2 Example of non-Anosov QAD in 3-dimensions

3 Results
Franks, Robinson example

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -6 & 5
\end{bmatrix}
\]

Let \( f_A \) be hyperbolic toral automorphism of \( \mathbb{T}^3 \) induced by \( A \).
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\( A \) has 1 stable eigenvalue and 2 real unstable eigenvalues.
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1. Let \( p \) be a fixed point and deform \( f_A \) in neighborhood of \( p \) to get \( f_1 \) a DA-diffeomorphism.
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2. Let \( f_2 = f_1^{-1} \) be diffeomorphism on \( \mathbb{T}^3 \). So there exists a repeller \( \Lambda_2 \) and 1-dimensional unstable foliation for \( \Lambda_2 \).
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1. Let \( p \) be a fixed point and deform \( f_A \) in neighborhood of \( p \) to get \( f_1 \) a DA-diffeomorphism.
2. Let \( f_2 = f_1^{-1} \) be diffeomorphism on \( \mathbb{T}^3 \). So there exists a repeller \( \Lambda_2 \) and 1-dimensional unstable foliation for \( \Lambda_2 \).
3. Take out neighborhood of \( p \) in each and perform surgery to attach. Adjusting so there is a quasi-transverse intersection.
Visualization of FR example - part 1

$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(M - V)$ is a codimension-one hyperbolic attractor.
\[ \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(M - V) \text{ is a codimension-one hyperbolic attractor} \]
Visualization of FR example - part 2
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3 Results
First Main Result

Theorem
(F., J. Rodriguez-Hertz) Let $f$ be a QAD of $M$ where $M$ is a closed orientable 3-manifold. Then the prime decomposition of $M$ is the connected sum of $k$ 3-tori possibly with handles. In case $M$ is non-orientable, then the tori are quotiented by involutions.
Second Main Result

Theorem

(F., J. Rodriguez-Hertz) Let $f$ be a QAD of a closed 3-manifold $M$. Then

1. The non-wandering set $\Omega(f)$ of $f$ consists of a finite number of codimension-one expanding attractors, codimension-one shrinking repellers and hyperbolic periodic points.

2. For each attractor $\Lambda$ in $\Omega(f)$, there exist a hyperbolic toral automorphism $A$ with stable index one, a finite set $Q$ of $A$-periodic points, and a linear involution $\theta$ of $T^3$ fixing $Q$ such that the restriction of $f$ to its basin of attraction $W^s(\Lambda)$ is topologically conjugate to a DA-diffeomorphism $f_A^Q$ on the punctured torus $T^3 - Q$ quotiented by $\theta$. In case $M$ is an orientable manifold, $\theta$ is the identity map. An analogous result holds for the repellers of $\Omega(f)$.
Outline of arguments

1. From results of Plykin ('80) we know that the basin of attraction for a codimension-one attractor is homeomorphic to a torus minus a finite set of points. (Note: Using these results of Plykin the first theorem is a consequence of the second.)
Outline of arguments

1. From results of Plykin ('80) we know that the basin of attraction for a codimension-one attractor is homeomorphic to a torus minus a finite set of points. (Note: Using these results of Plykin the first theorem is a consequence of the second.)

2. Using this structure we show that if $\Lambda$ is a codimension-one attractor and $\Lambda_0$ is a basic set with stable dimension one where $W^u(\Lambda_0) \cap W^s(\Lambda) \neq \emptyset$, then $\Lambda_0$ is a periodic orbit.
Outline of arguments

1. From results of Plykin ('80) we know that the basin of attraction for a codimension-one attractor is homeomorphic to a torus minus a finite set of points. (Note: Using these results of Plykin the first theorem is a consequence of the second.)

2. Using this structure we show that if Λ is a codimension-one attractor and Λ₀ is a basic set with stable dimension one where \( W^u(Λ₀) \cap W^s(Λ) \neq \emptyset \), then Λ₀ is a periodic orbit.

3. This implies that, if the diffeomorphism is not Anosov, all attractors and repellers of 3-dimensional QAD are codimension-one and all other basic sets consist of periodic orbits.
Definition

A point $p$ in a codimension-one attractor $\Lambda$ is a boundary point if there is a component of $W^s(p) - \{p\}$ that does not intersect $\Lambda$.

Boundary points can be shown to be periodic and occur in pairs of the same period such that if $p_1$ and $p_2$ are paired periodic points and if $x \in W^u(p_1)$, then there exists a point $y \in W^u(p_2) \cap W^s(x)$ and arc $(x, y)_s$ in $W^s(x)$ such that $(x, y)_s \cap \Lambda = \emptyset$. 
Proof of step 2 (cont)

We show there if Λ₀ is codimension-one basic set such that \( W^u(Λ₀) ∩ W^s(Λ) \), then there is a periodic point \( q \) in \( Λ₀ \) and paired boundary points \( p_1 \) and \( p_2 \) such that \( W^u(q) \) intersects an arc. We then get a fundamental domain of \( W^u(q) \) in \( W^s(Λ) \). Which implies that \( Λ₀ \) is periodic orbit.
We construct an example where $\Omega(f)$ consists of codimension-one attractors and repellers and some periodic orbits (modification of Franks and Robinson).

**Proposition**

If $f : M \to M$ is QAD of 3-manifold and $M \neq \mathbb{T}^3$, (not Anosov), then $f$ is approximated by QAD that are $\Omega$-conjugate to $f$, but not topologically conjugate to $f$.
Additional Results

We construct an example where $\Omega(f)$ consists of codimension-one attractors and repellers and some periodic orbits (modification of Franks and Robinson).

**Proposition**

If $f : M \to M$ is QAD of 3-manifold and $M \neq \mathbb{T}^3$, (not Anosov), then $f$ is approximated by QAD that are $\Omega$-conjugate to $f$, but not topologically conjugate to $f$.

**Theorem**

(F., J. Rodriguez-Hertz) If $f : M \to M$ is QAD of 3-manifold and $f$ is partially hyperbolic, then $f$ is Anosov.

Partially hyperbolic means $TM = E^u \oplus E^c \oplus E^s$. In dimension 3 we could state the above for a dominated splitting.
Partial hyperbolicity result

Note: In the preprint we assume dynamical coherence ($E^c$ is integrable).

- We show this implies there is a codimension-one foliation without Reeb components.
- This implies the manifold is irreducible.
- This then implies $M = \mathbb{T}^3$ so then Anosov.
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Dynamical coherence can be removed by recent result of Burago and Ivanov. Says that there are smooth foliations as near $E^c$ as one wants. This then implies again no Reeb components.
Open problems

1. If $M$ is a compact invariant submanifold for an Anosov diffeomorphism $f$, is $f|_M$ Anosov?
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2. If $M$ is a torus and $f$ is QAD on $M$, is $f$ Anosov? (Note: This is conjectured by Mañé.)
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3. If $f$ is QAD and partially hyperbolic (dominated splitting), is $f$ Anosov?
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4. If $M$ is 4-dimensional and $f$ is QAD on $M$, what can be said about structure of $M$ and action of $f$? (There are partial results by J. Rodriguez-Hertz, Ures, and Vieitez.)
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