Robustly transitive diffeomorphisms

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Dynamical systems

The setting for a dynamical system is:
▶ a space,
▶ a time, and
▶ a time evolution.

The space is called a **phase space** \((X \text{ or } M)\) and represents the possible states of the system. The time is either **continuous** \((\mathbb{R})\) or **discrete** \((\mathbb{Z})\). The time evolution is given by a **flow** \((\Phi_t)\) in the continuous case and a **map** \((f)\) in the discrete case.
2 main concepts

In studying \((X, f)\) there are 2 tools that are frequently used.

1. Look for \textbf{invariant sets}. That is \(\Lambda \subset X\) such that \(f(\Lambda) = \Lambda\).

2. Decompose \(X\) into invariant sets \(\Lambda_1, ... \Lambda_k\) and \textbf{wandering components} (i.e. points that under a forward orbit approaches one of the invariant sets and under backward iterates approaches another set).
Transitivity

**Definition:** A system \((X, f)\) is transitive if there exists some \(x \in X\) such that \(\mathcal{O}^+(x)\) is dense in \(X\) (where \(\mathcal{O}^+(x) = \{f^n(x) | n \in \mathbb{N}\}\)).

**Remark**

*This implies that there is a decomposition of the space into one invariant set and no wandering components.*
Transitivity

Definition: A system \((X, f)\) is transitive if there exists some \(x \in X\) such that \(O^+(x)\) is dense in \(X\) (where \(O^+(x) = \{f^n(x) | n \in \mathbb{N}\}\)).

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Example: Let \(X = S^1\) and \(f_\alpha(x) = (x + \alpha) \mod 1\). This is called a circle rotation. An exercise, is that if \(\alpha\) is irrational, then \((S^1, f_\alpha)\) is transitive. In fact, in this case every point has a forward orbit that is dense.
Another definition of transitivity

Proposition

Let $f : X \to X$ be continuous and $X$ be a compact metric space. Then $f$ is transitive if and only if for any open set $U$ and $V$ in $X$ there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.

Note: One direction is trivial.

To see the other let $\{V_i\}_{i=1}^{\infty}$ be a basis for the topology. Then $\bigcup_{n \in \mathbb{N}} f^{-n}(V_i)$ is dense in $X$ for each $i$ since it intersects every open set. So $Y = \bigcap_{i \in \mathbb{N}} (\bigcup_{n \in \mathbb{N}} f^{-n}(V_i))$ is a dense set of points. If $y \in Y$, then for each $i$ we know $y \in f^{-n}(V_i)$ for some $n$ and $f^n(y) \in V_i$. Then $\mathcal{O}^+(y)$ is dense in $X$. 
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**Remark**

$X$ need only be locally compact and Hausdorff for the above proposition.
Topology on the set of diffeomorphisms

Let $M$ be a compact, smooth, boundaryless, connected manifold. We let $\text{Diff}(M)$ be the set of diffeomorphisms from $M$ to $M$. (A diffeomorphism is a continuous bijection with where $Df(x)$ and $Df^{-1}(x)$ are defined and invertible for all $x \in M$.)

We want to define a metric on this set. For $f, g \in \text{Diff}(M)$ we set

$$d_1(f, g) = \sup_{x \in M} \{d(f(x), g(x)), \|Df(x) - Dg(x)\|\}.$$ 

It is not hard to see that this defines a metric. So $f$ and $g$ are close if points are moved uniformly close to one another and the derivatives stay close.

Note: With this topology the set $\text{Diff}(M)$ is an infinite dimensional metric space.

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Robust Transitivity
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Robustly transitive diffeomorphisms

**Definition:** A diffeomorphism $f$ is **robustly transitive** if there exists a neighborhood $\mathcal{U}$ of $f$ in $\text{Diff}(M)$ such that each $g \in \mathcal{U}$ is transitive.

**Remark**

A circle rotation $f_\alpha$ is transitive if $\alpha$ irrational, but $f_\alpha$ is not robustly transitive. To see this notice that if $\alpha$ is rational, then every point in $S^1$ is periodic under $f_\alpha$. (i.e. $f_\alpha^n(x) = x$ for some $n \in \mathbb{N}$ and all $x \in S^1$.)
Comments on robust transitivity

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2. The dynamics of these systems are very interesting. There is a great deal of complexity seen in these systems that is still not well understood. In 2-dimensional systems there is a good understanding of the systems, but in higher dimensions there is still much that is unknown.
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2. The dynamics of these systems are very interesting. There is a great deal of complexity seen in these systems that is still not well understood. In 2-dimensional systems there is a good understanding of the systems, but in higher dimensions there is still much that is unknown.

3. Although the condition is topological the theory is closely related to stable (robust) ergodicity a measurable (and probabilistic) concept.
Hyperbolic periodic points

To show robust transitivity we will see it is useful to look at hyperbolic periodic points.

Definition
A periodic point \( p \) of period \( n \) is **hyperbolic** if \( Df^n p \) has no eigenvalues on the unit circle.
Hyperbolic periodic points

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Definition

A periodic point \( p \) of period \( n \) is hyperbolic if \( Df^n p \) has no eigenvalues on the unit circle.

Remark:

- \( T_p M = E^s \oplus E^u \) where each vector \( v \in E^s \) is uniformly exponentially contracted under \( Df^n \) and each vector \( v \in E^u \) is uniformly exponentially expanded under \( Df^n \).
- \( W^s(p) = \{x \in M : d(f^{in}(x), f^{in}(p)) \to 0, i \to \infty\} \) is an immersed copy of Euclidean space and tangent at \( p \) to \( E^s \). (Similarly, for \( E^u \) using \( f^{-1} \).)
Basic transitivity criterion

**Lemma**

If $f$ has a hyperbolic periodic point $p$ whose stable and unstable manifolds are (robustly) dense in $M$, then $f$ is (robustly) transitive.
Example - hyperbolic toral automorphisms

The first known examples of robustly transitive diffeomorphisms were hyperbolic toral automorphisms.

**Definition:** An $n \times n$ matrix $A$ is a **hyperbolic toral automorphism** if it has

1. non-negative integer entries,
2. determinant 1, and
3. all eigenvalues lie off the unit circle.
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3. all eigenvalues lie off the unit circle.

**Remark:** Any such $A$ induces a diffeomorphism of the $n$-torus, denoted $f_A$. For $f_A$ one can show that the stable and unstable manifolds of the origin are robustly dense in the $n$-torus.
Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix has determinant 1 and two eigenvalues $\lambda_1 = (3 + \sqrt{5})/2 > 1$ and $\lambda_2 = 1/\lambda_1$. The eigenvectors of $A$ are

$$v_{\lambda_1} = ((1 + \sqrt{5})/2, 1) \text{ and } v_{\lambda_2} = ((1 - \sqrt{5})/2, 1).$$

Since the slopes are irrational we know that the projection of subspaces spanned by eigenvectors are dense. So the origin has stable and unstable manifolds that are dense in the 2-torus.

Remark: Using a condition called structural stability the transitivity can be shown to be robust.
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Other examples

In the 1970’s, Shub and Mañé constructed examples of non-hyperbolic robustly transitive diffeomorphisms.

► Although, these are not the hyperbolic toral automorphisms both of these examples were constructed as $C^0$ perturbations of hyperbolic toral automorphisms.

► The idea is to change the dynamics, quite drastically, in a neighborhood, but do this in such a way that the basic transitivity criterium still holds.

In the 1990’s new examples were given by Bonatti and Díaz, and Bonatti and Viana. These new examples are in many respects much more complicated.
Consequences of robust transitivity

Theorem

(Bonatti, Díaz, Pujals, and Ures) Every robustly transitive diffeomorphism has a weakened form of hyperbolicity (called volume hyperbolicity).

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**Theorem**
(Díaz and F.) There is a $C^1$-residual set of certain non-hyperbolic robustly transitive diffeomorphisms that have no symbolic extensions.
Let \( A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \) (This construction can be carried out by more general hyperbolic toral automorphisms) and \( F_0 : \mathbb{T}^4 \to \mathbb{T}^4 \) be defined as \( F_0(x, y) = (A^2x, Ay) \).

**Remark:** So the expansion and contraction in the \( x \)-direction are stronger (dominate) the contraction and expansion in the \( y \)-direction.
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We now want to change the map. Let \( F(x, y) = (A^2 x, f_x(y)) \) where \( f_x \in \text{Diff}(\mathbb{T}^2) \) and depends smoothly on \( x \). Such a map is called a **skew product**. The space \( \mathbb{T}^2 \times \{0\} \) is called the **base** and each \( \{x\} \times \mathbb{T}^2 \) is called a **fiber**.
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**Remark:** We will add conditions to \( f_x \).
Let $p$ be a fixed point for $A$ and $q$ be a different fixed point for $A^2$. Let $U$ be a very small neighborhood in $\mathbb{T}^2$ containing $q$. (So it will not contain $p$.) Let $f_x = A$ for $x \notin U$.

At $q$ we want $f_q$ to be a DA-diffeomorphism where the perturbation is done about $(q, p)$. (I will describe a DA-diffeomorphism later.)
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Then Shub shows that $W^s(\{p\} \times \mathbb{T}^2)$ is dense in $\mathbb{T}^2$ since the stable manifold of $p$ is dense in the base for $A^2$. Also, $W^s(p)$ is dense in $\mathbb{T}^2$ under the action of $A$ so $W^s(p, p)$ is dense in $\mathbb{T}^4$. $F$ is not hyperbolic (Anosov) since there are periodic points of different indices.
Shub example - part 2

Let $p$ be a fixed point for $A$ and $q$ be a different fixed point for $A^2$. Let $U$ be a very small neighborhood in $\mathbb{T}^2$ containing $q$. (So it will not contain $p$.) Let $f_x = A$ for $x \notin U$.

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By the domination of $A^2$ over $A$ this can be shown to be robust (i.e. - using what is called normal hyperbolicity).
DA -diffeomorphisms

Perturb $A$ in neighborhood of $p$. This is done so the stable direction remains unchanged, but at $p$ the stable direction becomes expanding ($p$ becomes a source). Then 2 new fixed points are formed with 1-stable and 1-unstable direction.
Open problems

Question

*Are there robustly transitive diffeomorphisms without hyperbolic periodic points?*

Remark:

In dimension 2 the only one is the torus. In dimension three can the 3-sphere?

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What dynamics are possible? Are all robustly transitive diffeomorphisms actually robustly topologically mixing?

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What are the mechanisms from which robust transitivity arises?
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