Super-exponential growth of the number of periodic orbits inside homoclinic classes

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Abstract

Joint work with Christian Bonatti and Lorenzo Diaz.
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We will show that $C^1$ generically if a homoclinic class contains periodic points of different indices, then super-exponential growth of periodic points inside the homoclinic class.
Definition: A diffeomorphism $f$ is Artin-Mazur (A-M) if the number of isolated periodic points of period $n$ of $f$, denoted $\text{Per}(n, f)$, grows exponentially fast.

There exists $C > 0$ such that $\#\text{Per}(n; f) \cdot \exp(Cn)$ for all $n \geq N$. Artin-Mazur (’65) proved A-M maps are dense in the space of diffeomorphisms.
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**Artin-Mazur diffeomorphisms**

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For diffeomorphisms generic sets are dense. Corresponds to topologically typical.
Kaloshin’s result

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Technical point in Kaloshin’s proof says an open set $\mathcal{K}$ of diffeomorphisms has super-exponential growth of periodic points generically if for every sequence of positive integers $a = \{a_n\}_{n=1}^{\infty}$ there is a generic subset $\mathcal{R}(a)$ of $\mathcal{K}$ such that

$$\limsup_{n \to \infty} \frac{\#\text{Per}(n, f)}{a_n} = \infty$$

for any $f \in \mathcal{R}(a)$.
Definition: For a hyperbolic periodic point $p$ of $f$ the *homoclinic class* $H(p) = \overline{W^s(O_p) \cap W^u(O_p)}$. Defined by Newhouse ('72) as generalization of hyperbolic basic sets. We will show generically the superexponential growth occurs inside a homoclinic class.
**Homoclinic Class**

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We will show generically the superexponential growth occurs inside a homoclinic class.
Consider a $C^1$ open set of diffeomorphism $\mathcal{U}$ such that each $f \in \mathcal{U}$ contains hyperbolic periodic saddles $p_f$ and $q_f$ depending continuously on $f$. From (ABCDW, preprint) there is a generic set $G$ of $\mathcal{U}$ such that either $H(p_f, f) = H(q_f, f)$ for all $f \in G$, or $H(p_f, f) \neq H(q_f, f)$ for all $f \in G$. First case called generically homoclinically linked, g.h.l.
Linked periodic points

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- either $H(p_f, f) = H(q_f, f)$ for all $f \in \mathcal{G}$,
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Super-exponential growth of the number of periodic orbits inside homoclinic classes – p. 7/22
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First case called \textit{generically homoclinically linked g.h.l.}
Note: Homoclinically related for periodic points says $W^s(p) \cap W^u(q)$ and $W^u(p) \cap W^s(q)$. This is an open condition, but is different than generically homoclinically linked.
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**Definition:** The index of a hyperbolic periodic point is the dimension of $E^s(p)$.

**Remark:** Homoclinically related periodic points have the same index, but generically homoclinically linked periodic points can have different indices.
Theorems

Theorem 1 (Bonatti, Diaz, F.) There is a residual subset $S(M)$ of $\text{Diff}^1(M)$ of diffeomorphisms $f$ such that, for every $f \in S(M)$ any homoclinic class of $f$ containing hyperbolic periodic saddles of different indices has super-exponential growth of the number of periodic points.
Theorem 1 (Bonatti, Diaz, F.) There is a residual subset $S(M)$ of $\text{Diff}^1(M)$ of diffeomorphisms $f$ such that, for every $f \in S(M)$ any homoclinic class of $f$ containing hyperbolic periodic saddles of different indices has super-exponential growth of the number of periodic points.

We can actually show if indices are $\alpha$ and $\beta$ with $\alpha < \beta$ then for each $\gamma \in [\alpha, \beta]$ there is super-exponential growth of periodic points of index $\gamma$ in the homoclinic class.
Corollary: Every non-hyperbolic homoclinic class of a $C^1$-generic diffeomorphism with a finite number of homoclinic classes has super-exponential growth of the number of periodic points.
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Remark: In (ABCDW) it is shown that $C^1$ generically the indices of saddles in the homoclinic class form an interval in $\mathbb{N}$. 
For a hyperbolic homoclinic class $\Lambda$ we know

$$h_{\text{top}}(f|\Lambda) = \limsup_{n \to \infty} \frac{\log \# P(n, f)}{n}.$$  

Then the periodic points have exponential growth equal to topological entropy.
Hyperbolic homoclinic class

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Then the periodic points have exponential growth equal to topological entropy.

In fact if $\Lambda$ is topologically mixing $\exists c_1, c_2 > 0$ such that

$$c_1 e^{nh_{\text{top}}(f|\Lambda)} \leq \# P(n, f) \leq c_2 e^{nh_{\text{top}}(f|\Lambda)}$$

for all $n$. 
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For $C^1$ diffeomorphisms this is equivalent to Axiom A plus no-cycles (Hayashi(’97) and Aoki(’92)).
**Star Diffeomorphisms**

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$C^1$ generic diffeomorphisms in complement of star diffeomorphisms have super-exponential growth, but not necessarily in homoclinic class.
Prevalence

There are examples (easy to construct) of generic sets with zero Lebesgue measure. Measure theoretically typical and topologically typical don’t always agree.

**Definition:** For a one parameter family of diffeomorphisms $f_t$ a property is *prevalent* if it holds on a set of full Lebesgue measure.
Prevalence

There are examples (easy to construct) of generic sets with zero Lebesgue measure. Measure theoretically typical and topologically typical don’t always agree.

**Definition:** For a one parameter family of diffeomorphisms $f_t$ a property is *prevalent* if it holds on a set of full Lebesgue measure.

**Problem:** (Arnold) For a (Baire) generic finite parameter family of diffeomorphisms $f_t$, for Lebesgue almost every $t$ we have that $f_t$ is A-M.
If there are g.h.l. points $p$ and $q$ of index $\alpha$ and $\alpha + 1$ respectively, then after a perturbation there is a saddle-node $r$ with $\dim(\mathbb{E}^s(r)) = n - \alpha - 1$, $\dim(\mathbb{E}^u(r)) = \alpha$, $\dim(\mathbb{E}^c(r)) = 1$.\"
If there are g.h.l. points \( p \) and \( q \) of index \( \alpha \) and \( \alpha + 1 \) respectively, then after a perturbation there is a saddle-node \( r \) with \( \dim(\mathbb{E}^s(r)) = n - \alpha - 1 \), \( \dim(\mathbb{E}^u(r)) = \alpha \), \( \dim(\mathbb{E}^c(r)) = 1 \).

Furthermore, \( W^s(r) \cap W^u(q) \neq \emptyset \) and \( W^u(r) \cap W^s(p) \neq \emptyset \).
Picture of $p, q, \text{ and } r$
Idea of proof

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Let \( a = \{a_n\}_{n=1}^{\infty} \) be a given sequence. Let \( r \) be of period \( n \). After perturbation \( r \) is identity in center direction so we can perturb to get \( na_n \) saddles of index \( \alpha \) (and \( na_n \) of index \( \alpha + 1 \))
Creation of periodic points
Idea of proof - part 2

For each $k \in \mathbb{N}$ there exists a residual set $G^\alpha(k)$ such that each $g \in G(k)$ there exists $n_g(k) \geq k$ where $H(p_g, g)$ has $n_g(k)a_{n_g(k)}$ different periodic points of index $\alpha$. 
Idea of proof - part 2

For each $k \in \mathbb{N}$ there exists a residual set $\mathcal{G}^\alpha(k)$ such that each $g \in \mathcal{G}(k)$ there exists $n_g(k) \geq k$ where $H(p_g, g)$ has $n_g(k)a_{n_g(k)}$ different periodic points of index $\alpha$.

Now let $\mathcal{R}^\alpha(a) = \bigcap_k \mathcal{G}^\alpha(k)$. This is residual.
Important Claim

Claim: For each $f \in \mathcal{R}^\alpha(a)$ it holds that

$$\limsup_{k \to \infty} \frac{\#\text{Per}^\alpha(f, k)}{a_k} = \infty.$$
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Idea of proof. Since $f \in \mathcal{G}_\alpha(k)$ for all $k \in \mathbb{N}$, there exists $n_f(k) \geq k$ such that $H(p, f)$ contains $n_f(k)a_{n_f(k)}$ points of index $\alpha$. So there is a subsequence $n_k$ such that

$$\frac{\#(\text{Per}_\alpha(f) \cap H(p_f, f))}{a_{n_k}} \geq n_k.$$
End of proof

Now let $\mathcal{R}(a) = \bigcap_{\alpha=\gamma}^{\beta} \mathcal{R}^\gamma(a)$. The set $\mathcal{R}(a)$ is residual in $\mathcal{U}$ and growth of saddles is lower bounded by $a$. 
Conjecture

Residually homoclinic classes depend continuously on diffeomorphism and number of homoclinic classes is locally constant. A diffeomorphism is *wild* if the number of homoclinic classes is locally infinite.
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**Conjecture:** There is a $C^1$ generic dichotomy for diffeomorphisms: either the homoclinic classes are hyperbolic or there is a super-exponential growth of the number of periodic points.
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Conjecture: There is a $C^1$ generic dichotomy for diffeomorphisms: either the homoclinic classes are hyperbolic or there is a super-exponential growth of the number of periodic points.

Finite number done by corollary
Symbolic Extensions

**Definition:** A diffeomorphism has a symbolic extension if there exists shift space $(\Sigma, \sigma)$ and factor map $\pi : \Sigma \rightarrow M$ such that $f \pi = \pi \sigma$. 
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**Question:** Among diffeomorphisms containing a homoclinic class with periodic points of differing index is there a \(C^1\)-residual set \(S\) such that for any \(f \in S\) does not have a symbolic extension?
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