4.7 Gaussian Quadrature

In general, all quad. rules are a particular case of
\[ \int_a^b f(x)dx = \sum_{i=1}^{n} \cdot C_i \cdot f(x_i) \]  

Important question: Best choices of \( C_i \)'s and \( x_i \)'s for higher degree of precision.

For example, for \( n=2 \): \( x_1, x_2 \)
and \( [a,b]=[\ -1,1] \), what should be \( x_1, x_2, a, c_2 \)
such that
\[ \int_{-1}^{1} f(x)dx = C_1 f(x_1) + C_2 f(x_2) \]  
is of highest degree of precision?

Since we want to determine 4 unknowns, it might be that degree of precision for an optimal choice is 3. It means we will try to determine \( x_1, x_2, c_1, c_2 \) such that (2) is exact for \( 1, x, x^2, \text{and } x^3 \).
We arrive to the following system of equations:

\[
\begin{align*}
C_1 + C_2 &= \int_{-1}^{1} dx = 2 \\
C_1 x_1 + C_2 x_2 &= \int_{-1}^{1} x dx = 0 \\
C_1 x_1^2 + C_2 x_2^2 &= \int_{-1}^{1} x^2 dx = \frac{x^3}{3} \bigg|_{-1}^{1} = \frac{2}{3} \\
C_1 x_1^3 + C_2 x_2^3 &= \int_{-1}^{1} x^3 dx = 0
\end{align*}
\]

Nonlinear system. Solutions are:

\[
C_1 = 1, \quad x_1 = -\frac{\sqrt{3}}{3} \\
C_2 = 1, \quad x_2 = \frac{\sqrt{3}}{3}
\]

Therefore, Quadrature rule using only two nodes with highest degree of precision is

\[
\int_{-1}^{1} f(x) dx \approx f \left( -\frac{\sqrt{3}}{3} \right) + f \left( \frac{\sqrt{3}}{3} \right).
\]  \( \text{(3)} \)

We can generate higher degree of precision formulas by following a similar procedure. But, there is an easier way.

- Notice that \( x_1 = -\frac{\sqrt{3}}{3} \) and \( x_2 = \frac{\sqrt{3}}{3} \) are the two roots of \( P_2(x) = x^2 - \frac{1}{3} \) (Legendre polyn. degree 2).
  They gave quad. formula exact to polynomials of degree less than 4.
- Show other Legendre polynomials (MAPLE).

In general, a quadrature formula exact for polynomials of degree less than an can be obtained using the roots $x_1, \ldots, x_n$ of Legendre polynomials of degree $n$. How about the coefficients $C_i$'s?

Theorem 4.7

1) $x_1, x_2, \ldots, x_n$ are roots of Legendre polynomial $P_n(x)$

2) The coefficients $C_i$, $i=1, \ldots, n$ are defined by

$$C_i = \int_{-1}^{1} \left( \prod_{j=1, \, j \neq i}^{n} \frac{(x-x_j)}{x_i-x_j} \right) dx$$

Then, if $P(x)$ is any polynomial of degree less than $2n$

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} C_i P(x_i)$$

"Exactly"

It means that the formula

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} C_i f(x_i)$$

is at least of degree of precision $2n-1$. 
Therefore, (3) is a Quad. Gauss formula with two nodes \((n=2)\) and degree of precision \(3\) \((2n-1)\).

The next formula, by using the previous thm is given by

\[
\int_{-1}^{1} f(x) dx = \frac{5}{9} f\left(-\sqrt[3]{5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt[3]{5}\right)
\]

\[
= 0.556 f(0.77) + 0.889 f(0) + 0.556 f(0.77)
\]

with degree of precision \(2(3)-1=5\).

**Gaussian Quadrature on any Interval**

Approx \(\int_{a}^{b} f(x) dx\) using Gauss quad.

I) Transform \([a,b] \rightarrow [-1,1]\)

\[x \rightarrow \frac{2}{b-a} (x-a)+1\]

Slope of \(L\) : \(m = \frac{2}{b-a}\)

\[t - 1 = \frac{2}{b-a} (x-b) \Rightarrow x-b = \frac{(b-a)(t-1)}{2}\]

or \[x = \frac{b-a}{2} (t-1) + b\], \(t \in [-1,1]\)

\[dx = \frac{b-a}{2} dt\]
Thus, changing variables \( x \to t \)

\[
\int_{a}^{b} f(x) \, dx = \int_{-1}^{1} f \left( \frac{b-a}{2} (t-1) + b \right) \left( \frac{b-a}{2} \right) \, dt
\]

In particular, for \( 1.5 = b \)

\[
\int_{1}^{1.5} e^{-x^2} \, dx
\]

\[
\int_{1}^{1.5} e^{-x^2} \, dx = \int_{-1}^{1} e^{-\left( \frac{t+5}{4} \right)^2} \frac{1}{4} \, dt 
\]

Thus, \( \int_{1}^{1.5} e^{-x^2} \, dx = \int_{-1}^{1} e^{-\left( \frac{t+5}{4} \right)^2} \frac{1}{4} \, dt 
\]

Gauss 2 points

\[
\approx -\frac{(\sqrt{3} + 5)^2}{16} - \frac{(\sqrt{3} + 5)^2}{16} = 0.1098003
\]

Gauss 3 points

\[
\approx \frac{5}{9} e \frac{16}{16} + \frac{8}{9} e \frac{16}{16} - \frac{5}{9} e \frac{(\sqrt{3} + 5)^2}{16}
\]

\[
\approx 0.1093642
\]
Table 4.11

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<tr>
<th>n</th>
<th>Roots $r_{n,i}$</th>
<th>Coefficients $c_{n,i}$</th>
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<td>0.5773502692</td>
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<tr>
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<td>-0.7745966692</td>
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Example 1  Approximate $\int_{-1}^{1} e^{x} \cos x \, dx$ using Gaussian quadrature with $n = 3$. The entries in Table 4.11 give us

$$
\int_{-1}^{1} e^{x} \cos x \, dx \approx 0.5 e^{0.7745966692} \cos 0.7745966692 \\
+ 0.8 \cos 0 + 0.5 e^{-0.7745966692} \cos(-0.7745966692)
$$
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The Gaussian quadrature procedure applied to this problem requires that the integral first be transformed into a problem whose interval of integration is $[-1, 1]$. Using Eq. (4.42) gives

$$
\int_{1}^{1.5} e^{-x^2} \, dx = \frac{1}{4} \int_{-1}^{1} e^{(-(t+5)^2/16)} \, dt.
$$

The values in Table 4.11 give the following Gaussian quadrature approximations for this problem:

**$n = 2$:**

$$
\int_{1}^{1.5} e^{-x^2} \, dx \approx \frac{1}{4} \left[ e^{-\left(5+0.5773502692\right)^2/16} + e^{-\left(5-0.5773502692\right)^2/16} \right] = 0.1094003;
$$

**$n = 3$:**

$$
\int_{1}^{1.5} e^{-x^2} \, dx \approx \frac{1}{4} \left[ (0.5555555556)e^{-\left(5+0.7745966692\right)^2/16} + (0.8888888889)e^{-\left(5\right)^2/16} 
\right.
\left. + (0.5555555556)e^{-\left(5-0.7745966692\right)^2/16} \right]
= 0.1093642.
$$