Consider a function $f$ such that

$$ f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R} \\
(x_1, \ldots, x_n) \to f(x_1, \ldots, x_n) $$  \hspace{1cm} (1)

0.1 Definition 1: Real-valued functions of $n$ variables

A function of $n$ variables is a rule that assigns a unique real number $f(x_1, \ldots, x_n)$ to each point $(x_1, \ldots, x_n)$ in $\mathcal{D}$.

0.2 Definition 2: Domain and Range of functions of $n$ variables

$f(x_1, \ldots, x_n)$ is called the image of $(x_1, \ldots, x_n)$ under $f$.

The set $\mathcal{D} \subset \mathbb{R}^n$ is called the domain of $f$.

The set of all real values $f(x_1, \ldots, x_n)$ is called the Range of $f$. It means

$$ \text{range of } f = \{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in \mathcal{D}\} $$

Discuss examples of functions of two variables.

The graph of a real-valued function $f$ of two is the variables is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in \mathcal{D}$.

0.3 Definition 3: Level Curves for functions of two variables

Assume the graph of a function $z = f(x, y)$ describes a surface $S$ in $\mathbb{R}^3$. For any real value $c$ in the range of $f$, the intersection of of $S$ with the plane $z = c$ is the curve

$$ f(x, y) = c $$

The projection of this curve onto the $xy$-plane is called a level curve of $f$. It is also called a contour curve.
Exercise: obtain level curves for
\[ z = f(x,y) = \ln (1 + x^2 + y^2) \]

Range of \( f = ? \), Domain = ?

Since \( 1 + x^2 + y^2 > 0 \) for all \((x,y) \in \mathbb{R}^2\)

and \( \ln (z) \) is defined for all \( z > 0 \)

Domain \( f = \mathbb{R}^2 \)

Range of \( f \) The function \( \ln(x) \) has the graph

\[
\begin{array}{c}
\text{graph} \\
\end{array}
\]

Clearly, range of \( \ln(x) \) is \( \mathbb{R} = (-\infty, \infty) \).

Notice that \( 1 + x^2 + y^2 > 1 \), \((x,y) \in \mathbb{R}^2\)

and \( \ln(x) \) for \( x > 1 \) is such that \( \ln x \in [0, +\infty) \)

So range of \( f(x,y) = \ln (1 + x^2 + y^2) \) is \([0, +\infty)\).
Level Curves:

\[ f(x,y) = c, \quad \text{for } C \in \text{range of } f. \]

Then for \( C > 0 \)

\[ \ln (x^2 + y^2 + 1) = c \]

What curves are these?

Hard to know!

However,

\[ e^c = e^c \]

\[ \ln (x^2 + y^2 + 1) \]

\[ x^2 + y^2 + 1 = e^c \quad \Rightarrow \quad x^2 + y^2 = e^c - 1 \]

For \( C > 0 \), these are circles of radius \( e^c - 1 \).