Math 314
Sections 05

MIDTERM 1

Name: ___________________________ Section: ___________________________

Feb 12-13, 16 2016

Answer all questions and show all your work carefully. Graphic Calculators are not allowed, but a regular scientific one will be permitted. There is a time limit for this test of 4 hours (much more than needed). Please Do NOT USE PEN. Use a regular pencil. Should you have need for more space than is allocated to answer a question, use the back of the page the problem is on and indicate this fact. Please do not talk about the test with other students until after the last day to take the exam.

…but if ye are prepared ye shall not fear.
D&C 38:30

Prof. Vianey Villamizar

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1. (a) For the curve \( C \)
\[
r(t) = e^t \cos(t)i + e^t \sin(t)j + e^t k,
\] (1)
which defines the position of a moving particle at time \( t \), answer the following questions:

i. (2 points) Sketch the curve with vector equation (1) and indicate with an arrow the direction in which \( t \) increases.

ii. (5 points) Find the arc distance (arc length) travelled by this particle from time \( t = 0 \) to \( t = 2\pi \).

iii. (3 points) Also, find the straight-line distance between the endpoints of the curve \( (r(0) \text{ and } r(2\pi)) \) described by the particle motion. Compare with the arc distance computed in the previous item. Explain.

(b) (6 points) For the curve \( C \):
\[
r(t) = (2 \cos t)i + (3 \sin t)j + \sqrt{5} \cos tk.
\]
Find a tangent vector and the tangent line at \( t = \pi/2 \).

\( 
\begin{align*}
\text{i)} & \quad \vec{r}'(t) = \langle \cos t - \sin t, \sin t + \cos t, 1 \rangle \\
\text{ii)} & \quad |\vec{r}'(t)| = e^t \sqrt{2 + 2 + 1} = e^t \\
\Rightarrow & \quad |\vec{r}'(0)| = \sqrt{3} e^0 = \sqrt{3} \\
\text{Arc length} & = \int_0^{2\pi} |\vec{r}'(t)| dt = \int_0^{2\pi} \sqrt{3} e^t dt \\
& = \sqrt{3} e^t \bigg|_0^{2\pi} = \sqrt{3} (e^{2\pi} - 1) \approx 925.77.
\end{align*}
\)

\( 
\begin{align*}
\text{iii)} & \quad \text{Straight-line distance:} \\
\vec{r}(0) & = \langle 1, 0, 1 \rangle, \quad \vec{r}(2\pi) = \langle 1, 0, 1 \rangle e^{2\pi} \\
\Rightarrow & \quad d = \sqrt{2(e^{2\pi} - 1)^2} \approx 755.88.
\end{align*}
\)

Explain: arc length \( > \) \( d \) as expected!. 
Straight path \( \neq \) Curved path \( \text{btw two points} \).
\( b) \quad \vec{r}(t) = \langle 2 \cos t, 3 \sin t, \sqrt{5} \cos t \rangle

Tangent vector: \( \vec{r}'(t) = \langle -2 \sin t, 3 \cos t, -\sqrt{5} \sin t \rangle \)

Tangent line at \( t = \pi/2 \):

\( \vec{r}'(\pi/2) = \langle -2, 0, -\sqrt{5} \rangle, \quad \vec{r}(\pi/2) = \langle 0, 3, 0 \rangle \)

\( L: \langle x, y, z \rangle = t \langle -2, 0, -\sqrt{5} \rangle + \langle 0, 3, 0 \rangle \)
2. For the function \( f(x, y) = \ln \left(\frac{y}{x^2}\right) \),

(a) (5 points) Determine the domain and the range (show your algebra work to obtain them)

(b) (3 points) Sketch the domain in the \( xy \)-plane. Carefully show those points in the plane that are in the domain and those which are not.

(c) (8 points) Identify the level curves \((f(x, y) = c)\) of the above function and sketch the level curves corresponding to \( c = -2, -1, 0, 1, 2 \) in the \( xy \)-plane.

(a) Domain: \( \frac{y}{x^2} > 0 \Rightarrow y > 0, x \neq 0 \) or \( \mathbb{D} = \{(x, y) : x \neq 0 \text{ and } y > 0\} \)

(b) Range:

Choose \( x = 1 \) and let \( y \) take any real value, \( y > 0 \)

then \( \ln \left(\frac{y}{x^2}\right) = \ln(y), \ y \in \mathbb{R} \)

\( \Rightarrow \ln(y) \in (-\infty, \infty) = \mathbb{R} \).

(c) \( \ln \left(\frac{y}{x^2}\right) = c \Rightarrow \frac{y}{x^2} = e^c \)

\( \Rightarrow y = e^c x^2, \ x \neq 0 \)
3. (a) (10 points) Find the limit, if it exists, or show that the limit does not exist.

\[ i) \quad \lim_{(x,y) \to (0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2} \]

\[ ii) \quad \lim_{(x,y) \to (2,0)} \frac{xy - 2y}{y^2 + (x - 2)^2} \]

Hint: For (b) study the limit at (2,0) along a line passing through the point (2,0) with slope 1. Then, consider any other direction. What is your conclusion?

(b) (5 points) Determine the set of points (sketch this set in the plane) at which the function

\[ f(x, y) = \sqrt{x} + \sqrt{1 - x^2 - y^2} \]

is continuous. Explain.

a) i) along \( y = mx \)

\[ \lim_{x \to 0} \frac{x^4 - 4m^2x^2}{2m^2x^2 + x^2} = \lim_{x \to 0} \frac{x^2(x^2 - 4m^2)}{x^2(2m^2 + 1)} = \frac{-4m^2}{2m^2 + 1} \]

Different limits along lines \( y = mx \) \( \Rightarrow \) DNE. Along \( y = x \), \( \lim_{x \to 0} \frac{-4}{3} = -\frac{4}{3} \)

ii) along \( x = 2 \)

\[ \lim_{y \to 0} \frac{2y - 2y}{y^2 + (x - 2)^2} = \lim_{y \to 0} \frac{0}{y^2} = 0. \]

Along \( y = x - 2 \)

\[ \lim_{(x, y) \to (2,0)} \frac{x(x - 2) - 2(x - 2)}{(x - 2)^2 + (x - 2)^2} = \lim_{(x, y) \to (2,0)} \frac{(x - 2)(x - 2)}{(x - 2)^2} \]

\[ = \lim_{(x, y) \to (0,0)} \frac{1}{2} = \frac{1}{2} \]
3) b) \[ f(x,y) = \sqrt{x} + \sqrt{1-x^2-y^2} \]

**Continuity:** It is defined when \[ x \geq 0, \quad 1-x^2-y^2 > 0 \Rightarrow x^2+y^2 < 1 \]

This function is continuous in its domain of definition.
- There is a limit at any \((x,y) \in D\).
- \( \lim_{(x,y) \to (x_0,y_0) \in D} f(x,y) = f(x_0,y_0) \).
4. (a) (6 points) Verify that the function \( u(x, t) = e^{x+3t} + \sin(x - 3t) \) is a solution of the wave equation \( u_{tt} = 9 u_{xx} \).

(b) (9 points) A surface \( S \) has a tangent plane at the point \( P(2, 1, 3) \), if the curves:
\[ r_1(t) = < 2 + 3t, 1 - t^2, 3 - 4t + t^2 > \text{ and } r_2(u) = < 1 + u^2, 2u^3 - 1, 2u + 1 > \]
both lie on \( S \). Verify that the point \( P(1, 2, 3) \) is in both curves and find an equation of the tangent plane at \( P \).

\[ 4) \ a) \quad U_t = 3 e^{x+3t} + 3 \cos(x-3t), \quad U_{tt} = 9 e^{x+3t} - 9 \sin(x-3t) \]
\[ U_x = e^{x+3t} - \cos(x-3t), \quad U_{xx} = e^{x+3t} - \sin(x-3t) \]

Clearly, \( U_{tt} = 9 U_{xx} \).

\[ \begin{align*}
&\text{b) } r_1'(0) = < 3, 1, 3 >, \quad r_2'(1) = < 2, 1, 3 > \quad \text{Verified} \\
&Tangent \text{ Vectors at } (1,2,3) \]
\[ r_1'(t) = < 3, 2t, -4+2t > \Rightarrow r_1'(0) = < 3, 0, -4 > \]
\[ r_2'(u) = < 2u, 6u^2, 2 > \Rightarrow r_2'(1) = < 2, 4, 2 > \]

\[ \hat{n}_{\text{plane}} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
3 & 0 & -4 \\
2 & 6 & 2
\end{vmatrix} = \begin{vmatrix} 4 \end{vmatrix} - 8 - 6, 18 = < 24, 14, 18 > \quad \text{or } < 12, 7, 9 > \]

Plane: \( < 12, 7, 9 > \cdot < x-2, y-1, z-3 > = 0 \)

\[ 12(x-2) + 7(y-1) + 9(z-3) = 0 \]

\[ 12x + 7y + 9z = 24 + 7 + 27 = 44 \]
5. (10 points) Assume that the vector functions \( \mathbf{u} \) and \( \mathbf{v} \) are differentiable. Prove the formula for the product rule of the dot product:

\[
\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)
\]

\[
\tilde{\mathbf{u}}(t) = < u_1(t), u_2(t), u_3(t) > \\
\tilde{\mathbf{v}}(t) = < v_1(t), v_2(t), v_3(t) >
\]

\[
\frac{d}{dt} [\tilde{\mathbf{u}}(t) \cdot \tilde{\mathbf{v}}(t)] = \frac{d}{dt} [u_1 v_1 + u_2 v_2 + u_3 v_3]
\]

\[
= u_1' v_1 + u_1 v_1' + u_2' v_2 + u_2 v_2' + u_3' v_3 + u_3 v_3'
\]

\[
= (u_1' v_1 + u_2' v_2 + u_3' v_3) + (u_1 v_1' + u_2 v_2' + u_3 v_3')
\]

\[
= \tilde{\mathbf{u}}' \cdot \tilde{\mathbf{v}} + \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}'
\]

6. (10 points) Show that the lines

\[
r(t) = <1, 1, 0 > + t <1, -1, 2 >
\]

\[
r(t) = <2, 0, 2 > + t <-1, 1, 0 >
\]

intersect at the point \( P(2, 0, 2) \) and find an equation of the plane that contains these lines.

**Intersection:**

For \( \mathbf{c}_1 \):

\[
\tilde{r}_1(t) = \langle 1, 1, 0 \rangle + t \langle 1, -1, 2 \rangle
\]

\[
\tilde{r}_1(1) = \langle 2, 0, 2 \rangle
\]

For \( \mathbf{c}_2 \):

\[
\tilde{r}_2(t) = \langle 2, 0, 2 \rangle + t < -1, 1, 0 >
\]

\[
\tilde{r}_2(0) = \langle 2, 0, 2 \rangle
\]

So they intersect at \( P(2, 0, 2) \).

**Equation of plane containing these two lines:**

\[
\tilde{\mathbf{n}} = \begin{vmatrix}
\hat{e}_1 & \hat{e}_2 \\
1 & -1 \\
-1 & 1 \\
\end{vmatrix} = \langle -2, -2, 0 \rangle
\]

Equation:

\[
\langle -2, -2, 0 \rangle \cdot \langle x-2, y, 2-2 \rangle = 0
\]

\[
2x + 2y = 4 \\
x + y = 2
\]
7. (18 points) Consider the quadric surfaces defined by the equations

i) \(-25x^2 + y^2 - 9z^2 - 1 = 0\),  
ii) \(3x^2 + \frac{1}{25}y^2 - (z-2)^2 = 0\) \((\text{shifted } +2 \text{ in } z \text{ direction})\)
iii) \(4x^2 + y^2/9 - z + 2 = 0\)  
iv) \(3x^2 + \frac{1}{25}y^2 - 1 = 0\) \((\text{ellipse in } \mathbb{R}^3 \text{ plane})\) \(\text{and for all } z\)

The two graphs below correspond to two of the equations above, match them. Make the graph (including scales along the axis) for the other two equations on the spaces provided. Include in the graphs their corresponding equations.

\((\text{i})\) \(3\sqrt[3]{x^2 + \frac{1}{25}y^2} - (z-2)^2 = 0\)
\((\text{a})\)

\((\text{iii})\) \(3x^2 + \frac{1}{25}y^2 - 1 = 0\)
\((\text{c})\)

\((\text{ii})\) \(-25x^2 + y^2 - 9z^2 = 1\)
- \(x = k \rightarrow y^2 - 9z^2 = 25k^2 + 1\) \((\text{hyperbola})\)
- \(y = k \rightarrow 75x^2 + 1 = k^2 - 1\) \((\text{ellipse})\) \(\text{undefined for } |z| > 1\)
- \(z = k \rightarrow y^2 - 75x^2 = 1 + 9k^2\) \((\text{hyperbola})\)

\((\text{iii})\) \(4x^2 + \frac{1}{9}y^2 - z = -2\)
- \(x = k \rightarrow \frac{1}{4}y^2 - z = -\frac{4k^2 - 2}{9}\) \((\text{hyperbola})\)
- \(y = k \rightarrow 4x^2 - z = -\frac{1}{9}y^2 - 2\) \((\text{hyperbola})\)
- \(z = k \rightarrow \frac{4}{9}x^2 + \frac{1}{9}y^2 = -2 + k\) \((\text{ellipsoid})\) \(\text{undefined for } k < 2\)