1. (10 points) Evaluate the line integral \( \int_C xyz \, ds \), where \( C \) is the curve given by \( x(t) = 2 \sin t, \ y(t) = 1, \) and \( z(t) = -2 \cos t, \) for \( 0 \leq t \leq \pi/4. \)

\[
I = \int_0^{\pi/4} x(t) y(t) z(t) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt
\]

\[
= \int_0^{\pi/4} 2 \sin t \cdot 1 \cdot (-2 \cos t) \sqrt{4 \cos^2 t + 4 \sin^2 t + 4 \sin^2 t} \, dt
\]

\[
= -4 \int_0^{\pi/4} \sin t \cos t \sqrt{4} \, dt = -8 \left. \frac{\sin^2 t}{2} \right|_0^{\pi/4} = -4 \left( \frac{\sqrt{2}}{2} \right)^2 = -2
\]
2. Consider the region $R$ bounded by

\[
x - y = 0, \quad x - y = \pi, \quad x + 2y = 0, \quad x + 2y = \frac{1}{2}\pi.
\]

(a) (3 points) Draw the region $R$ given above (1) in the $xy$-plane. Include scales along the axis.

(b) (4 points) Evaluate the following integral on the region $R$ defined above (1) by making an appropriate change of variables.

\[
\iint_{R} \sin(x - y) \cos(x + 2y) \, dx \, dy,
\]
1) Domain $R$ in the $x,y$-plane is bounded by:

\[ x - y = 0, \quad x + y = \pi, \quad x + 2y = 0, \quad x + 2y = \frac{\pi}{2} \]

\[ I = \iint_R \sin(x - y) \cos(x + 2y) \, dx \, dy \]

a) \[ \text{Wrong limits} \]

b) \[ I = \int_0^\pi \int_0^{\pi/2} \sin(u) \cos(v) \frac{1}{3} \, dv \, du \]

\[ J(u, v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{vmatrix} = \frac{2}{3} + \frac{1}{3} = \frac{2}{3} = \frac{1}{3} \]

\[ = \frac{1}{3} \int_0^\pi \sin u \, \sin v \, \cos u \, \cos v \, du \]

\[ = \frac{1}{3} \int_0^\pi \sin u \, \cos u \, du = \frac{1}{3} \left[ \frac{1}{2} \cos^2 u \right]_0^\pi = \frac{1}{3} \left( \frac{1}{2} \right) = \frac{2}{3} \]

b) Area of $R$:

\[ \iint_R \, dx \, dy = \int_0^\pi \int_0^{\pi/2} \left| J(u, v) \right| \, dv \, du = \frac{1}{3} \pi \cdot \frac{\pi}{3} = \frac{\pi^2}{9} \]
3) (10 points) Integrate the vector field $\mathbf{h}(x,y) = (y^2, 2xy - e^{2y})$ over the circular arc $C: \mathbf{r}(t) = (\cos t, \sin t)$ where $0 \leq t \leq \pi/2$.

First, let's explore if $\mathbf{h}(x,y)$ is a conservative vector field. Why? Because, this will simplify the calculations.

Now,

$$\frac{\partial Q}{\partial x} = \partial y = \frac{\partial P}{\partial y} \tag{3}$$

Therefore $\mathbf{h}$ is conservative.

If we can find a potential function $f$ such that

$$\mathbf{h}(x,y) = \nabla f(x,y)$$

then

$$\int_C \mathbf{h} \cdot d\mathbf{r} = f(\text{end point}) - f(\text{initial point}). \tag{*}$$

Finding $f$:

If there is such $f$ then

$$f_x = y^2 \quad \text{and} \quad f_y = 2xy - e^{2y}$$

Integrating (1) w/ respect to $x$,

$$f(x,y) = xy^2 + C(x, y)$$

then

$$f_y = 2xy + \frac{\partial C}{\partial y} \tag{2}$$

Combining this equal w/ (2)

$$2xy + \frac{\partial C}{\partial y} = 2xy - e^{2y} \Rightarrow \frac{\partial C}{\partial y} = -e^{2y} \quad \text{then integrating w/ respect to } y$$

$$C(y) = -\frac{e^{2y}}{2} + d$$

Substituting into (3)

$$f(x,y) = xy^2 - \frac{e^{2y}}{2} + d \tag{5}$$

Now, we can use (*)&

Initial point $= \mathbf{r}(0) = \langle 1, 0 \rangle$

End point $= \mathbf{r}(\pi/2) = \langle 0, 1 \rangle$

Then

$$\int_C \mathbf{h} \cdot d\mathbf{r} = f(0,1) - f(1,0)$$

$$= -\frac{1}{2}e^2 - \left(-\frac{1}{2}\right) = \frac{1}{2}(1 - e^2)$$

Note: We cannot apply Green's Theorem because $C$ is not closed.
3. (10 points) Integrate the vector field \( h(x, y) = (y^2, 2xy - e^{2y}) \) over the circular arc
\( C: r(t) = (\cos t, \sin t) \) where \( 0 \leq t \leq \pi/2. \)
\[ h(r(t)) = (\sin^2 t, 2\cos t \sin t - e^{2\sin t}) \]
\[ d\mathbf{r} = (-\sin t, \cos t) \]
\[ \int_C h \cdot d\mathbf{r} = \int_C \frac{r'(t) \cdot r''(t)}{|r'(t)|} \, dt \]
\[ = \int_C F \cdot r'(t) \, dt \]
\[ = \int_0^{\pi/2} (-\sin^2 t + 2\sin t \cos^2 t - \cos t e^{2\sin t}) \, dt \]
\[ = -\left( \frac{2}{3} \cos^3 t + \frac{1}{2} e^{2\sin t} \right) \bigg|_0^{\pi/2} - \int_0^{\pi/2} \sin^2 t \, dt \]
\[ = -\left( 0 + \frac{1}{2} e^2 - \frac{2}{3} - \frac{1}{2} e^0 \right) - \int_0^{\pi/2} \sin^2 t \, dt \]
\[ = \frac{7}{6} - \frac{1}{2} e^2 - \int_0^{\pi/2} \sin^2 t \, dt \]
\[ = \frac{7}{6} - \frac{1}{2} e^2 - \int_0^{\pi/2} \sin(1 - \cos^2 t) \, dt \]
\[ \text{Let } u = 1 - \cos^2 t, \quad v = -\cos t \]
\[ du = 2\sin t \cos t \, dt, \quad dv = \sin t \, dt \]
\[ = \frac{7}{6} - \frac{1}{2} e^2 - \left( (1 - \cos^2 t)(-\cos t) \right) \bigg|_0^{\pi/2} + \int_0^{\pi/2} 2\sin t \cos^2 t \, dt \]
\[ = \frac{7}{6} - \frac{1}{2} e^2 - \left( (\cos^2 t - \cos t) \right) \bigg|_0^{\pi/2} + \left( -\frac{2}{3} \cos^3 t \right) \bigg|_0^{\pi/2} \]
\[ = \frac{7}{6} - \frac{1}{2} e^2 - \left( 0 - 0 - 1 - 0 + \frac{2}{3} \right) \]
\[ = \frac{7}{6} - \frac{1}{2} e^2 - \frac{2}{3} = \frac{1}{2} - \frac{1}{2} e^2 \]