L: \[ \langle x, y, z \rangle = t \langle 1, 1 \rangle + \langle 1, 3 \rangle \]

Line passing through points \((1, 3)\) and \((4, 6)\)

Slope \(m = \frac{6 - 3}{4 - 1} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3}{3} = 1\)

Line \(y - 3 = 1(x - 1)\) or \[y = x + 2\]

This derivation of the equation of a line in the plane cannot be extended to the space.

Alternative:  ① Find a vector in the same direction of the line \(L\)

Show \(\vec{v} = \langle 4, 6 \rangle - \langle 1, 3 \rangle = \langle 11, 7 \rangle\).

Obtain points in \(L\) by adding a multiple of \(\vec{v}\) \(t\vec{v}\) to the position vector of \(P_0(1, 3)\).

Any \((x, y) \in L\) satisfies \(\langle x, y \rangle = t \langle 1, 1 \rangle + \langle 1, 3 \rangle\).
12.5 **Lines and Planes.**

**Vector form of a line** that passes through \( P_0(x_0, y_0, z_0) \) with direction \( \vec{v} = \langle a, b, c \rangle \)

To every point \( P(x, y, z) \) on \( L \) there is a vector \( \langle x, y, z \rangle \) associated.

Clearly, by adding vectors

\[
\langle x, y, z \rangle = t\langle a, b, c \rangle + \langle x_0, y_0, z_0 \rangle, \quad t \in \mathbb{R} \tag{1}
\]

If \( \vec{v} = \langle x, y, z \rangle \) and \( \vec{v}_0 = \langle x_0, y_0, z_0 \rangle \)

then,

\[
\vec{v} = \vec{v}_0 + t\vec{v}, \quad t \in \mathbb{R} \tag{2}
\]

(1) or (2) are the line \( L \) vector equations.

**Remark:** For any point on \( L \), there is a \( t \in \mathbb{R} \) such that (1) is satisfied. Conversely, for any \( t \) there is a \( P(x, y, z) \in L \).

**Parametric form of line \( L \).**

\[\langle x, y, z \rangle = t\langle a, b, c \rangle + \langle x_0, y_0, z_0 \rangle \] is equivalent to

\[
x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad t \in \mathbb{R} \tag{3}
\]
Symmetric Equations of a Line $L \ (a, b, c \neq 0)$

From (3)

$$t = \frac{x-x_0}{a}, \quad t = \frac{y-y_0}{b}, \quad t = \frac{z-z_0}{c}$$

then

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

(4)

Discuss:

i) Parallel lines,  ii) Perpendicular lines  

iii) Skew lines.

Show Maple worksheet on lines.

Line segment $\overline{L}$ between $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$

$$\vec{r}_0 \rightarrow \vec{v}_o \quad \vec{r}_1 \rightarrow \vec{v}_1$$

$$\overrightarrow{P_0P_1} = \vec{v}_1 - \vec{v}_0 \quad \text{direction vector}$$

Line through $P_0$ with $\overrightarrow{P_0P_1}$ as dir. vector

$$\vec{v} = \vec{v}_0 + t(\vec{v}_1 - \vec{v}_0) \iff \vec{v} = (1-t)\vec{v}_0 + t\vec{v}_1$$

If $t=0 \Rightarrow \vec{v} = \vec{v}_0$, if $t=1 \Rightarrow \vec{v} = \vec{v}_1$

Any point betwe $P_0$ and $P_1$ can be reached for a specific $0 < t < 1$. Therefore, 
Equ. for segment

$$\vec{v} = (1-t)\vec{v}_0 + t\vec{v}_1, \quad 0 \leq t \leq 1$$
Planes

- Ask student for equations for planes.

Planes through \( P_0 \)? How many?

Planes parallel to \( \vec{v} \)?

Planes perpendicular to \( \vec{n} \)? Add one point to define a plane

Planes perpendicular to \( \vec{n} \) and passing through \( P_0 \)?

\[ \vec{r} = \vec{OP}, \quad \vec{r}_0 = \vec{OP}_0 \]

\[ \vec{P} \rightarrow \vec{r}, \quad P_0 \vec{P} = \vec{r} - \vec{r}_0 \]

\[ (\vec{r} - \vec{r}_0) \cdot \vec{n} = 0 \]

Therefore,

\[ \vec{r} = \vec{OP} = \langle x, y, z \rangle, \quad \vec{r}_0 = \vec{OP}_0 = \langle x_0, y_0, z_0 \rangle \]

\[ \langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0 \]

or

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]  
Scalar equ.

or

\[ ax + by + cz + d = 0 \]  
Linear equ.

When \( d = -(ax_0 + by_0 + cz_0) \)
Discuss:

i) Parallel planes

ii) Angle between planes is the acute angle btw their normals. \( \hat{n}_1 \cdot \hat{n}_2 \) \[ \cos \theta = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1| |\hat{n}_2|} \]

iii) Perpendicular planes.

iv) Intersection of two planes. \( \rightarrow \) line \( L \) or entire plane if same.

v) Equation of a plane passing through three points.

If \( \hat{a} = \hat{P}\hat{Q} \)
\[ \hat{b} = \hat{P}\hat{R} \]

Then, \( \hat{n} = \hat{a} \times \hat{b} \)

Thus, using \( \hat{n} \) and \( P, Q, \) or \( R \) the plane eqn can be obtained.
#33) Find eqn. of plane thru \( P (2,1,2) \)
\( Q (3,-8,6) \)
\( R (-2,3,1) \)
\[
\vec{PQ} = (1,-9,4)
\vec{PR} = (-4,-4,-1)
\]
\[
\vec{PQ} \times \vec{PR} = \vec{n} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & -9 & 4 \\
-4 & -4 & -1
\end{vmatrix} = \hat{i}(25) - \hat{j}(15) + \hat{k}(-40)
\]
\[
= (25,-15,-40) = 5(5,3,-8)
\]

Using \( \vec{n} = (5,3,-8) \) and \( P_0 (2,1,2) \).

\[
\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \Rightarrow 5x-3y-8z = 10-3-16 = -9
\]
\[
5x-3y-8z = -9
\]
12.5 #48  Line through \( L: (-3,1,0) \) and \((1,5,6)\)

Find where \( L \) intersects \( \Pi: 2x+y-2z=-2 \).

**Answer:**

\[ \vec{u} = \vec{P_0P_1} = \langle 2, 4, 6 \rangle \]

\[ L: \langle x, y, z \rangle = t \langle 2, 4, 6 \rangle + \langle -3, 1, 0 \rangle. \]

\[ \vec{n} \parallel \Pi. \quad \vec{n} = \langle 2, 1, -1 \rangle \]

If \( \vec{n} \parallel \vec{u} \) there is not intersection.

However, \( \vec{u} \cdot \vec{n} = \langle 2, 4, 6 \rangle \cdot \langle 2, 1, -1 \rangle = 2 \neq 0. \)

So, there is an intersection.

To find it, we realize that points on \( L \) satisfy

\[ x = 2t - 3, \quad y = 4t + 1, \quad z = 6t \]

then, for \((x, y, z)\) to be in the plane

\[ 2(2t-3) + 4t+1 - 6t = -2 \]

\[ 2t - 6 + 1 = -2 \Rightarrow t = \frac{3}{2} \]

and \( P^* (0, 7, 9) \)

\[ \langle x^*, y^*, z^* \rangle = \frac{3}{2} \langle 2, 4, 6 \rangle + \langle -3, 1, 0 \rangle = \langle 3 - 3, 7, 9 \rangle \]

\[ = \langle 0, 7, 9 \rangle. \]
Problem

(Similar to (12))

Find the line of intersection of the two planes

$\Pi_1: x + y + z = -1$, $\Pi_2: 2x - 3y + 2z = 2$

Answer: For the line we need

a) a direction vector $\vec{v} = \langle a, b, c \rangle$.

b) a point on the line. $P_0(x_0, y_0, z_0)$

1. To obtain a direction vector $\vec{v}$

$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 1, 1, 1 \rangle \times \langle 2, -3, 2 \rangle$

$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -3 & 2 \end{vmatrix} = 5\hat{i} + 0\hat{j} - 5\hat{k} = \langle 5, 0, -5 \rangle$

2. To obtain a point on $L$

$x = 0 \Rightarrow (y + 2 = -1)(+3)$

$3y + 2z = 2$

$0 + 0z = \Rightarrow z = -\frac{1}{5} \Rightarrow y = -1 - 2 = -\frac{7}{5}$

or $\langle x_0, y_0, z_0 \rangle = \langle 0, -\frac{1}{5}, -\frac{7}{5} \rangle$

Then $L: \langle x, y, z \rangle = t \langle 5, 0, -5 \rangle + \langle 0, -\frac{1}{5}, -\frac{7}{5} \rangle$
We can also find a 2nd point $P_2(x_2, y_2, z_2)$ at the intersection of $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$(0, -4/5, -1/5)$

2nd point: $z = 0 \quad (x + y = -1) (2)$

$2x - 3y = 2$

$-5y = 4 \Rightarrow y = -4/5 \Rightarrow x = -1/5$

$\Rightarrow \quad P_2(x_2, y_2, z_2) = P_2(-1/5, -4/5, 0)$

And the line at the intersection is

$t \langle -\frac{1}{5}, 0, \frac{1}{5} \rangle + \langle 0, -\frac{4}{5}, -\frac{1}{5} \rangle = \langle x, y, z \rangle$

Same as $t \langle 5, 0, -5 \rangle + \langle 0, -\frac{4}{5}, -\frac{1}{5} \rangle = \langle x, y, z \rangle$

Why?

$\langle -\frac{1}{5}, 0, \frac{1}{5} \rangle (-25) = \langle 5, 0, -5 \rangle$

Either one can be used as a direction vector.

Another alternative $x = t$

$\left\{ \begin{array}{l}
(y + 2 = -1-t) (2) \\
-3y + 2z = 2 - 3t
\end{array} \right.$

$-5y = 4 \Rightarrow y = -4/5$

$z = -1-t + 4/5$

$\langle x, y, z \rangle = \langle t, -4/5, -1-t + 4/5 \rangle$

$= t \langle 1, 0, -1 \rangle + \langle 0, -4/5, -1/5 \rangle$. 
Find the plane that passes thru $P_0 (3,1,4)$ and contains a line $L$, intersection of the two planes

\[
\begin{align*}
3x + 2y + 3z &= 1 \\
2x - y + 2z &= -3
\end{align*}
\]

Strategy: 0. Obtain $\vec{v}$ using the two planes

\[\vec{v} = \hat{n}_1 \times \hat{n}_2\]

1. Find equation for $L$. Common to $P_1 (x_1, y_1, z_1)$ both planes.

2. Find a point on $L$, $P_1 (x_1, y_1, z_1)$ both planes.

3. Form a vector $\vec{w}$ on the plane sought $\vec{w} = \vec{P_0} P_1$, $P_0 (3,1,4)$.

4. Obtain normal $\hat{n}$ of plane sought as $\hat{n} = \vec{v} \times \vec{P_0} P_1$.

Allow: 0. \(\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \langle 5, 5, -5 \rangle\)

1. $L: \langle x, y, z \rangle = t \langle 5, 5, -5 \rangle + \langle 0, 3, -1 \rangle$.

2. Point on $L$ $x = 0, 2y + 3z = 1$ $2 (-y + 2z = -3)$ 0. \[z = -1 \Rightarrow y = -1 + 3 \Rightarrow y = 2\]

4. $\vec{w} = \langle 3, 1, 4 \rangle - \langle 0, 3, -1 \rangle = \langle 3, -1, 5 \rangle$

5. $\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 5 & 5 \\ 5 & 5 & 3 \end{vmatrix} = \langle 20, 40, -20 \rangle$ or $\langle -1, 2, -1 \rangle$
#73) Find the distance between the parallel planes

\[ \Pi_1 : 2x - 3y + z = 4 \quad \text{and} \quad \Pi_2 : 4x - 6y + 2z = 3 \]

Ans. To compute the distance, we need to:

a) Obtain a perpendicular line to both planes

b) Determine the two points in which this line \( L \) intersects the two planes.

![Diagram showing perpendicular line to two planes]

(c) Compute the distance between \( P_0 \) and \( P_1 \).

First, a point in \( \Pi_1 \) is \( <0,0,4> \)

Second, a line perpendicular to \( \Pi_1 \) and \( \Pi_2 \) through \( <0,0,4> \) is \( L : \langle x, y, z \rangle = t \langle 2, -3, 1 \rangle + <0,0,4> \).

Third, intersection with plane \( \Pi_2 \):

Same as intersection of a given line with a plane.
L: x = 2t, y = -3t, z = t + 4, parametric form

Substituting into eqn for \( T_2 \)

\[
4(2t) - 6(-3t) + 2(t + 4) = 3
\]

\[
8t + 18t + 2t + 8 = 3 \Rightarrow 28t = -5 \Rightarrow t = -\frac{5}{28}
\]

\[
\Rightarrow x = -\frac{5}{14}, \ y = \frac{15}{28}, \ z = \frac{-5}{28} + 4 = \frac{112 - 5}{28} = \frac{107}{28}
\]

Therefore, point of intersection of line L with plane \( T_2 \)

\[
\left( -\frac{5}{14}, \frac{15}{28}, \frac{107}{28} \right)
\]

Forth: Distance from \( (0, 0, 4) \) to \( \left( -\frac{5}{14}, \frac{15}{28}, \frac{107}{28} \right) \) in \( T_2 \)

\[
d = \sqrt{\frac{25}{14^2} + \frac{(15)^2}{28^2} + \left( \frac{107}{28} - 4 \right)^2}
\]

\[
= \sqrt{\frac{25}{196} + \frac{225}{784} \left( \frac{5}{28} \right)^2} = \sqrt{\frac{25}{196} + \frac{225}{784} + \frac{25}{784}}
\]

\[
= \sqrt{\frac{25}{196} + \frac{250}{784}} = \sqrt{\frac{25}{196} + \frac{125}{392}} = \sqrt{\frac{50 + 125}{392}}
\]

\[
= \sqrt{\frac{175}{392}} = \frac{5\sqrt{7}}{28} \approx 0.6682.
\]