16.2 Line Integrals

Integration along a curve \( \mathcal{C} \).

Consider a curve \( \mathcal{C} \) in \( \mathbb{R}^2 \)

\[ x = x(t), \quad y = y(t), \quad a \leq t \leq b. \]

or equivalently

\[ \mathcal{C} : \mathbf{r}(t) = \langle x(t), y(t) \rangle \]

We assume \( \mathcal{C} \) is smooth, which means

\( \mathbf{r}'(t) \) is cont. and \( \mathbf{r}'(t) \neq \mathbf{0} \).

We divide the interval of \( t \) \([a, b]\) into \( n \) subintervals of equal width.

\[ x_i = x(t_i), \quad y_i = y(t_i) \]

The corresponding points \( P_i(x_i, y_i) \) divide \( \mathcal{C} \) into \( n \) subarcs with lengths \( \Delta s_1, \ldots, \Delta s_i, \ldots, \Delta s_n \).
$P_i^*$ is in $i^{	ext{th}}$ subarc
and corresponds to $t_i^* \in [t_{i-1}, t_i]$.

Now, if $f(x,y)$ is a function of two variables
whose domain includes $C$, we form the
analogous of a Riemann sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta S_i$$

Then, we consider the

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta S_i \quad (2.1)$$

If this limit exists, we use the notation

$$\int_C f(x,y) ds \quad (2.2)$$

And this limit is called

Line Integral of $f$ along $C$. 
Since,

\[ S(t) = \int_a^t \sqrt{x'(t)^2 + y'(t)^2} \, dt \]

\[ \Rightarrow \frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} \]  

length of \( C \) between \( a \) and \( t \).

Thus, we have the following result

If \( f(x, y) \) is a continous function, then the limit (3.1) always exists and we have the formula:

\[ \int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt \]

The value of the integral does not depend on the parametrization of the curve.

Notice, from (3.1)

\[ ds = |\hat{r}'(t)| \, dt = \sqrt{x'(t)^2 + y'(t)^2} \, dt \]

Add line integrals in space.
16.2 #12)

\[ I = \int_{0}^{1} (3x + 9z) \, ds, \quad \vec{c} = x(t) = t, \quad y(t) = t^2, \quad z(t) = t^3 \quad 0 \leq t \leq 1. \]

\[ I = \int_{0}^{1} \sqrt{(1)^2 + (2t)^2 + (2t^2)^2} \, dt \]

\[ = \int_{0}^{1} \sqrt{2t^3 + 9t^4} \, dt \]

If \( u = 1 + 4t^2 + 9t^4 \) \( \Rightarrow du = (8t + 36t^3) \, dt \)

or \( du = 4(2t + 9t^3) \, dt \)

Thus,

\[ I = \frac{1}{4} \int_{u=1}^{u=14} u^{-\frac{1}{2}} \, du = \frac{1}{4} \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{1}^{14} = \frac{1}{2} \left[ (14)^{\frac{3}{2}} - 1 \right] \]
16.2 # 9) (Discuss this in class if time allows)

\[ I = \int_C xyz \, ds \]

\[ \mathbf{r}(t) = \begin{cases} x(t) = 2 \sin t \\ y(t) = t \\ z(t) = -2 \cos t \end{cases}, \quad 0 \leq t \leq \pi \]

\[ I = -\int_0^\pi 4t \sin t \cos t \sqrt{4 \cos^2 t + 1^2 + 4 \sin^2 t} \, dt \]

\[ = -4 \int_0^\pi t \sin t \cos t \sqrt{5} \, dt = -\frac{2 \sqrt{5}}{5} \int_0^\pi t \cos(2t) \, dt \]

Now, \[ \int t \sin(2t) \, dt = -t \cdot \cos(2t) + \frac{1}{2} \int \cos(2t) \, dt \]

\[ \int t \sin(2t) \, dt = -t \cdot \cos(2t) + \frac{1}{2} \sin(2t) \]

Thus,

\[ I = -2 \sqrt{5} \left[ \left. -t \cos(2t) \right|_0^\pi + \frac{1}{2} \sin(2t) \right|_0^\pi \right] = \]

\[ = -2 \sqrt{5} \left[ \left. -\frac{\pi}{2} + \frac{1}{4} \cdot 0 \right] = \sqrt{5} \pi \right] = \]
If \( \mathbf{c} \) is piecewise-smooth, it means
\[
\mathbf{c} = c_1 \cup c_2 \cup \ldots \cup c_n \quad \text{and init. point of } c_{i+1} \text{ is final point of } c_i
\]
and each \( c_i \) is smooth, then
\[
\int_{\mathbf{c}} f(x,y,z) \, ds = \int_{c_1} f \, ds + \ldots + \int_{c_n} f \, ds.
\]

Properties:
\[
\int_{-\mathbf{c}} f(x,y) \, ds = \int_{\mathbf{c}} f(x,y) \, ds
\]

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Line Integrals of \( f(x,y,z) \) with respect to \( x, y, \) and \( z \).

Definition
\[
\int_{\mathbf{c}} f(x,y,z) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta x_i
\]

Partition along \( x \)-axis.

Analogously,
\[
\int_{\mathbf{c}} f(x,y,z) \, dy = \lim_{n \to \infty} \sum f(x) \Delta y_i
\]
\[
\int_{\mathbf{c}} f(x,y,z) \, dz = \lim_{n \to \infty} \sum f(x) \Delta z_i
\]
Using a parametric representation of $\mathcal{C}$:

\[
\int_{a}^{b} f(x,y,z) \, dx = \int_{a}^{b} f(x(t),y(t),z(t)) \times v(t) \, dt
\]

\[
\int_{a}^{b} f \, dy = \int_{a}^{b} f(\cdot) \, y'(t) \, dt
\]

\[
\int_{a}^{b} f \, dz = \int_{a}^{b} f(\cdot) \, z'(t) \, dt
\]

A combined integration w.r.t. $x$ and $y$ or w.r.t. $x, y, \text{and} z$

frequently appears.

It means

\[
I = \int_{\mathcal{C}} P(x,y) \, dx + \int_{\mathcal{C}} Q(x,y) \, dy = \int_{\mathcal{C}} P(x,y) \, dx + Q(x,y) \, dy
\]

Notice
\[ \int_{-b}^{b} f(x,y) \, dx = - \int_{-c}^{c} f(x,y) \, dx \]

This is because \( x(t) \) change sign.

**Line Integrals of Vector Fields.**

**In 1-d**

If \( f(x) \) is force acting on particle moving from \( x=a \) to \( x=b \).

\[ \text{Work} = \int_{a}^{b} f(x) \, dx. \]

We also know,

If \( \vec{F} \) (\text{const}) is a force in 3-d, then the work done to move an object from a point \( P \) to \( Q \) is given by

\[ W = \vec{F} \cdot \vec{PQ} = \vec{F} \cdot \vec{D}. \]
If we have a force given by the vector field
\[ \vec{F}(x, y, z) = P(x, y, z) \hat{i} + Q(x, y, z) \hat{j} + R(x, y, z) \hat{k} \]

Continuous force field in \( \mathbb{R}^3 \).

Want to compute work done by this force \( \vec{F} \) moving a particle along a smooth curve \( \mathcal{C} \).

Show P.P. graph with partition on \( \mathcal{C} \).

\[ \text{Work} = W = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*) \right] \Delta s_i \]

If this limit exists,
\[ = \int_{\mathcal{C}} \vec{F}(x, y, z) \cdot \vec{T}(x, y, z) \, ds = \int_{\mathcal{C}} \vec{F} \cdot \vec{T} \, ds. \]

If \( \mathcal{C} : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b \)
\[ \Rightarrow \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \quad \text{and} \]
\[ W = \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \, |\vec{r}'(t)| \, dt = \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]
**Definition**

A vector field \( \mathbf{F} \) is defined on a smooth curve \( C \) given by \( \mathbf{r}(t) \), \( a \leq t \leq b \). Then, the "Line Integral of \( \mathbf{F} \) along \( C \)"

\[
\oint_C \mathbf{F} \cdot d\mathbf{s}
\]

**Example**
Alternative Vector Fields:

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F}(x,y,z) \cdot \mathbf{T}(x,y,z) \, ds = \]
\[ \text{Lunit tang. Vector of } C \]

\[ = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \]

If \( \mathbf{F}(x,y,z) = P(x,y,z) \mathbf{i} + Q(x,y,z) \mathbf{j} + R(x,y,z) \mathbf{k} \)

And \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \)

Then, \( \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \)

\[ = \int_a^b \left[ P(x(t), y(t), z(t)) x'(t) + Q(x(t), y(t), z(t)) y'(t) + R(x(t), y(t), z(t)) z'(t) \right] \, dt \]

\[ = \int_a^b P(x(t), y(t), z(t)) x'(t) \, dt + \ldots + \int_a^b R(x(t), y(t), z(t)) z'(t) \, dt = \]

\[ = \int_C P(x,y,z) \, dx + \ldots + \int_C R(x,y,z) \, dz . \]