sect 1.7 #34 (true or false)

we know by hypothesis

1) \( \hat{v}_1 \) and \( \hat{v}_2 \) in \( \mathbb{R}^d \). It can be done exactly the same for \( \mathbb{R}^n \).

2) \( \hat{v}_2 \) not scalar multiple of \( \hat{v}_1 \).

Then \( \{\hat{v}_1, \hat{v}_2\} \) is linearly independent.

Answer: False. Counter-example

\( \hat{v}_1 = \hat{0} \) and \( \hat{v}_2 \neq \hat{0} \).

If we ask \( \hat{v}_1 \neq \hat{0} \) and \( \hat{v}_2 \neq \hat{0} \), then the statement is true.

We prove this as follows:

P: \( \hat{v}_2 \) not scalar multiple of \( \hat{v}_1 \),  Q: \( \{\hat{v}_1, \hat{v}_2\} \) lin. indep.

\[ 7Q \Rightarrow 7P. \]

If \( \{\hat{v}_1, \hat{v}_2\} \) lin. dep. then there exists \( c_1 \neq 0 \), \( c_2 \neq 0 \) such that

\[ c_1 \hat{v}_1 + c_2 \hat{v}_2 = \hat{0}, \]

\[ \Rightarrow c_2 \hat{v}_2 = -c_1 \hat{v}_1 \Rightarrow \hat{v}_2 = \frac{-c_1}{c_2} \hat{v}_1 \]

\[ \Rightarrow \hat{v}_2 = C^* \hat{v}_1, \quad \text{where} \quad C^* = \frac{-c_1}{c_2}. \]
(We know)

1) $\vec{v} = \vec{0}$, and $\vec{b}$ a vector in $\mathbb{R}^n$ not specified.

2) $L$ is the line given in parametric form by

$$L = \{ \vec{x} \in \mathbb{R}^n \text{ such that } \vec{x} = \vec{b} + t\vec{v}, \ t \in \mathbb{R} \}$$

If $n=2$

$$\begin{array}{c}
\overrightarrow{\vec{v}} \\
\overrightarrow{\vec{v}}
\end{array}$$

3) $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation

Then

The line $L$ is mapped into another line $L'$

$$L' = \{ \vec{x} \in \mathbb{R}^n \text{ such that } \vec{x} = \vec{q} + s\vec{u}, \ s \in \mathbb{R} \}$$

Proof:

$$\begin{array}{c}
\overrightarrow{\vec{v}} \\
\overrightarrow{\vec{v}}
\end{array} \quad \xrightarrow{T} \quad \begin{array}{c}
\overrightarrow{\vec{v}} \\
\overrightarrow{\vec{v}}
\end{array}$$

We want to show that any $\vec{x} \in L$ is such that $T(\vec{x}) \in L'$

for appropriate $\vec{q}$ and $\vec{u}$.
So, consider any $\tilde{x} \in L$, then $\tilde{x} = \tilde{p} + t \tilde{v}$, for certain $t \in \mathbb{R}$. Then, using linearity of $T$.

$$T(\tilde{x}) = T(\tilde{p} + t \tilde{v}) = T(\tilde{p}) + t T(\tilde{v}) = \tilde{q} + t \tilde{u}$$

Where $\tilde{q} = T(\tilde{p})$ and $\tilde{u} = T(\tilde{v})$.

Therefore, any $\tilde{x} \in L$ is such that $T(\tilde{x}) \in L'$, where

$$L' = \left\{ \tilde{x} \in \mathbb{R}^n \text{ such that } \tilde{x} = \tilde{q} + t \tilde{u} \right\}$$

Where $\tilde{q}$ and $\tilde{u}$ are defined as above.
Sect 1.8 # 33

\[ T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \]

\[ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \quad T(\mathbf{x}) = \begin{bmatrix} x_1 - 3x_2 \\ x_1 - 3 \\ 2x_1 - 5x_2 \end{bmatrix} \]

is not linear

**Proof:** We already proved that if \( T \) is linear \( T(\mathbf{0}) = \mathbf{0}. \)

Since \( \mathbf{0} = a\mathbf{u} \), for any \( \mathbf{u} \in \mathbb{R}^n. \)

then \( T(\mathbf{0}) = T(a\mathbf{u}) = aT(\mathbf{u}) = a\mathbf{0} = \mathbf{0}. \)

In this particular case,

\[ T(\mathbf{0}) = \begin{bmatrix} 0 - 2(0) \\ 0 - 3 \\ 2(0) - 5(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \neq \mathbf{0} \]

Therefore, \( T \) is not linear.

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**Hypothesis**

1) \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) linear, \( 2) \{ \mathbf{u}, \mathbf{v} \} \) lin. indep. in \( \mathbb{R}^n. \)

3) \( \{ T(\mathbf{u}), T(\mathbf{v}) \} \) lin. dependent in \( \mathbb{R}^m. \)

Then, \( T(\mathbf{x}) = \mathbf{0} \) has a nontrivial soln.

**Proof:** Want to prove that there is \( \mathbf{x}^* \in \mathbb{R}^n \) such that \( T(\mathbf{x}^*) = \mathbf{0}. \)

Consider a linear combination of \( T(\mathbf{u}), T(\mathbf{v}) \) where the scalars \( c_1^* = 0 \) or \( c_2^* = 0, \) but

\[ \text{Linearity} \quad c_1^* T(\mathbf{u}) + c_2^* T(\mathbf{v}) = \mathbf{0} \quad \text{(Possible based upon)} \]

\[ \text{Condition} \quad T(c_1^* \mathbf{u} + c_2^* \mathbf{v}) = \mathbf{0}. \]

Then \( \mathbf{x}^* = \frac{c_1^* \mathbf{u} + c_2^* \mathbf{v}}{\| \mathbf{v} \|} \quad \text{and} \quad T(\mathbf{x}^*) = \mathbf{0}. \)
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\[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

\[ T(v) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} \]
Some good problems of Chapter 1 of lay 4th edition

Sect 1.5
# 40

We know

1) $A_{3 \times 3}, b \in \mathbb{R}^3$

2) $Ax = b$, does not have a solution.

Want to know if

a vector $\hat{y} \in \mathbb{R}^3$ such that

$A\hat{x} = \hat{y}$ has a unique solution?

If 1 and 2, then by thm 2, the augmented matrix

\[
\begin{bmatrix}
A & 0
\end{bmatrix}
\]  

when row reduced echelon form

has a row as

\[
\begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & \text{?} \\
0 & 0 & 0 & \ldots & 0 & \text{?} \\
0 & 0 & 0 & \ldots & 0 & \text{?}
\end{bmatrix}
\]

Then, the reduced row echelon form of $A$ looks like

\[
\begin{bmatrix}
1 & 0 & * \\
0 & 1 & * \\
0 & 0 & 0
\end{bmatrix}
\]  

\[
\begin{bmatrix}
1 & * & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]  

\[
\begin{bmatrix}
1 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Therefore, if we consider the augmented matrix $[A \mid b]$

we find that its reduced row echelon form has $x_3$

as free variable in case 1. In case 2, $x_2$ is a free variable,

and in case 3 $x_2$ and $x_3$ are free variables.
Since these three cases are the only possibilities for $A$ under the conditions of the theorem. We conclude that there is not any $y \in \mathbb{R}^3$, such that $A\hat{x} = \hat{y}$, and $\hat{x}$ is unique.
Thm.- If $A$ is square matrix $n \times n$, $\text{Ann}_n$. Then,

$p: A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$ if and only if

$q: A$ has a pivot position in every row.

Proof.-

$p \Rightarrow q$

If $p$ is true, then the reduced row echelon form of the augmented matrix $[A \mathbf{b}]$ does not have a row as $[0 \ 0 \ \cdots \ 0 \ 0 \ \cdots \ 0 \ d]$, for any $\mathbf{b} \in \mathbb{R}^n$ (thm.2).

Therefore, the reduced row echelon form of $A$ does not have a row of zeros. In fact, it looks like

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
$$

Therefore, the matrix $A$ has a pivot position in every row.

(*) Thus, every row will have a nonzero entry. It can be easily seen that the only possibility for a reduced-row echelon form of a square matrix $A$, is to have $1$ in the diagonal entry as shown above.
If $Q$ is true, the only possible form for the reduced-row echelon form of $A$ is

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

Therefore, there are no free variables for the linear system whose augmented matrix is given by $[A \ | \ b]$ for every $b \in \mathbb{R}^n$. Thus, using Thm 2, the solution of $A\tilde{x} = \tilde{b}$ for any $b \in \mathbb{R}^n$ is unique.