CHAPTER 3

DETERMINANTS.

Reconsider the computation of the inverse of

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

"A is invertible iff \( ad - bc \neq 0 \)"

The quantity \( ad - bc \) is called \( \text{det}(A) \) and it's also denoted as \( \text{det}(A) \). It means

\[ \text{det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \]

Minors and Cofactors

To extend the concept of determinant to higher order square matrices, it's convenient to define

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

And then

\[ \text{det}(A) \overset{\text{def}}{=} a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \]
Definition of $\det(A)$ for a 3x3 matrix from the definition of $\det(A)$ for a 2x2 matrix.

Consider

$A = \begin{bmatrix} \frac{1}{1} & -4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{bmatrix}$

$M_{11} = \begin{vmatrix} -4 & 3 \\ -7 & -2 \end{vmatrix} = 14 - 21 = -7$

$A = \begin{bmatrix} -2 & 6 \\ 1 & -2 \end{bmatrix}$, $M_{12} = \begin{vmatrix} -2 & 6 \\ 1 & -2 \end{vmatrix} = 4 - 6 = -2$

$A = \begin{bmatrix} -4 & 3 \\ -2 & 6 \end{bmatrix}$, $M_{13} = \begin{vmatrix} -4 & 3 \\ -2 & 6 \end{vmatrix} = -14 + 7 = -7$

Observe

$a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} = 1(-7) - 4(-2) - 3(-7) = -20 + 21 = 1$

Definitions: $M_{11}, M_{12}, M_{13}$ are called minors

a) $M_{11}$ is the minor of entry $a_{11}$ and so forth...

b) We also define $C_{11} = (-1)^{1+1} M_{11} = M_{11}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$, and so forth.

Then, $a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} = 1$
Also, \[ A = \begin{bmatrix} 4 & -3 \\ -2 & 6 \\ 1 & 7 \\ 1 & -2 \end{bmatrix}, \quad M_{21} = \begin{vmatrix} 4 & -3 \end{vmatrix} = 13 \]
\[ A = \begin{bmatrix} 4 & -3 \\ -2 & 6 \\ 1 & 7 \\ 1 & -2 \end{bmatrix} \quad \Rightarrow \quad M_{31} = \begin{vmatrix} 4 & -3 \end{vmatrix} = 3 \]

And \[ a_{11} M_{11} - a_{21} M_{21} + a_{31} M_{31} = 1(-28) - (-2)(13) + 1(3) = 1 \]

Also, \[ M_{22} = \begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} = -2 + 3 = 1 \]
\[ M_{23} = \begin{vmatrix} 1 & 4 \\ 1 & 7 \end{vmatrix} = 7 - 4 = 3 \]
\[ M_{32} = \begin{vmatrix} 1 & -3 \\ -2 & 6 \end{vmatrix} = 6 - 6 = 0 \]
\[ M_{33} = \begin{vmatrix} 1 & 4 \\ -2 & -7 \end{vmatrix} = -7 + 8 = 1 \]

Also, \[ a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} = 1 \]

Where \[ C_{i1} = (-1)^{i+1}, \quad i = 1, 2, 3. \]
The Common number "1" obtained independently of the row or column considered is called \( \text{det}(A) \).

Therefore,

\[
\text{det}(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 1 \quad (2.1)
\]

The numbers \( c_{11}, c_{12}, c_{13} \) are called cofactors of their respective entries \( a_{11}, a_{12}, a_{13} \).

But also,

\[
\text{det}(A) = a_{11}c_{11} + a_{21}c_{21} + a_{31}c_{31} = 1 \quad (2.2)
\]

The expression \((2.1)\) is called cofactor expansion of 
A along 1\(^{st}\) row.
and \((2.2)\) correspond to a cofactor expansion of \( A \)
along 1\(^{st}\) column.
Also, \textit{2\textdegree\, Row} \\
\[-A_{21} \, M_{21} + A_{22} \, M_{22} - A_{23} \, M_{23} =
\]
\[= 2(13) + (-4)(1) - 6(3) = 26 - 7 - 18 = 1.
\]
\textit{3\textdegree\, Row} \\
\[A_{31} \, M_{31} - A_{32} \, M_{32} + A_{33} \, M_{33} =
\]
\[= 1(3) - 7(0) + (-2)(1) = 1
\]
Similarly, the corresponding combination for columns

\textit{Column 2} \\
\[-A_{12} \, M_{12} + A_{22} \, M_{22} - A_{32} \, M_{32} = 1
\]

\textit{Column 3} \\
\[A_{13} \, M_{13} - A_{23} \, M_{23} + A_{33} \, M_{33} = 1
\]

This common number \textquotedblleft 1\textquotedblright\ will be called the\[\text{det}(A), \text{ i.e., } \text{det}(A) = 1\]

\textbf{Definition:} the determinants \(M_{ij}(i, j = 1, 2, 3)\) of the \(2\times2\) matrices that remain after the \(i\text{th}\, row\) and \(j\text{th}\, column\) are deleted from \(A_{3\times3}\) are called the \textit{minor} of entry \(a_{ij}\).

The number \((-1)^{i+j} M_{ij}\) is called the \textit{Cofactor} of entry \(a_{ij}\).

The property observed in the previous computation for this particular \(3\times3\) matrix is true in general for any \(3\times3\) matrix \(A\).
This is the content of the following theorem:

Thus:- If \( A \) is a 3x3 matrix, then regardless of which row or column of \( A \) is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

This property can be easily proved for any 2x2 matrix. In fact, for \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \)

\[
M_{11} = a_{22}, \quad M_{12} = a_{21}, \quad M_{21} = a_{12}, \quad M_{22} = a_{11}
\]

\[
C_{11} = a_{22}, \quad C_{12} = -a_{21} = -M_{12}, \quad C_{21} = -a_{12} = -M_{21}, \quad C_{22} = a_{11}
\]

and

\[
A_{11}C_{11} + A_{12}C_{12} = A_{11}a_{22} - A_{12}a_{21} \quad \text{(1st row)}
\]

\[
A_{11}C_{11} + A_{21}C_{21} = A_{11}a_{22} - A_{21}a_{12} \quad \text{(1st column)}
\]

Similar for 2nd row and 2nd column. Therefore, the previous theorem is also verified by 2x2 matrices.
Def.: If $A$ is a $3 \times 3$ matrix, then the common number obtained by multiplying the entries in any row or column of $A$ by the corresponding cofactors and adding the resulting products is called the determinant of $A$, and the sum themselves are called cofactor expansions of $A$. That is,

$$\det(A) = a_{ij} C_{ij} + a_{kj} C_{kj} + a_{3j} C_{3j}, \quad \text{for any column} \quad j = 1, 2, 3$$

(Cofactor expansion along $j^{th}$ column)

And

$$\det(A) = a_{i1} C_{i1} + a_{iz} C_{iz} + a_{iz} C_{iz}, \quad \text{for any row} \quad i = 1, 2, 3$$

(Cofactor expansion along $i^{th}$ row)

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**Definition of $\det(A)$ for an $n \times n$ matrix**

The previous definition about minors and cofactors can be easily extended to $4 \times 4$ and inductively to any $n \times n$ matrix. Also, the theorem remains true for the $n \times n$ matrices. Therefore the definition of $\det(A)$ for an $n \times n$ matrix is a straightforward extension of the one for $3 \times 3$ matrices.

Given above. More specifically,

$$\det A = a_{ij} C_{ij} + a_{kj} C_{kj} + \cdots + a_{nj} C_{nj}, \quad \text{for any column} \quad j = 1, 2, \ldots, n$$

$$\det A = a_{i1} C_{i1} + a_{iz} C_{iz} + \cdots + a_{in} C_{in}, \quad \text{for any row} \quad i = 1, 2, \ldots, n.$$
Computing \( \text{det}(A) \) by a smart choice of Row or Column.

2.1) \#21)

\[
A = \begin{bmatrix}
-3 & 0 & 7 \\
2 & 3 & 1 \\
-1 & 0 & 5
\end{bmatrix}
\]

Choose 2\(^{nd}\) column, then

\[
\text{det}(A) = 0 \cdot c_{12} + 5 \cdot c_{22} + 0 \cdot c_{33} =
\]

\[
= 5 \cdot M_{22} = 5 \begin{vmatrix}
-3 & 7 \\
-1 & 5
\end{vmatrix} =
\]

\[
= 5 \cdot (-15 + 7) = -40.
\]

Arrow technique for 3x3 matrices

\# 11)

\[
A = \begin{bmatrix}
1 & 4 & -2 \\
3 & 5 & -1 \\
1 & 6 & 1
\end{bmatrix}
\]

\[
= (-20 - 7 + 92) - (20 + 84 + 6) =
\]

\[
= 45 - 110 = -65
\]

Same result that will be obtained by applying Cofactor expansion. See proof in book.

Determinant of an upper Triangular matrix

\[
\text{det} \left( \begin{bmatrix}
a_{11} & 0 & 0 \\
a_{21} & a_{22} & 0 \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} \right) = a_{11} a_{22} a_{33} = a_{11} a_{22} a_{33}
\]

Inductively can be proven that

\[
\begin{vmatrix}
a_{11} & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix} = a_{11} a_{22} \cdots a_{nn}.
\]
Consider
\[ A = \begin{bmatrix}
1 & 0 & 0 \\
3 & 2 & 0 \\
4 & 5 & 6
\end{bmatrix} \]
Then, 
\[ \det A = 1 \begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 4 & 5 \end{vmatrix} + 0 \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 1 \times 2 \times 6 = 12. \]
This result is true for any matrix \( A_{n \times n} \).

**Theorem 2.** If 
\[ A = \begin{bmatrix}
a_{11} & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & \cdots & \cdots & a_{nn}
\end{bmatrix} \]
then, 
\[ \det A = a_{11} a_{22} \cdots a_{nn}. \]
Show in class

\[ n = 2 \quad \vdash \quad n = \nu \]

\[ a_{11} \ldots a_{\nu \nu} = a_{11} \ldots a_{\nu \nu} \]

\[ a_{kh} \]

\[ 0 \]

\[ \begin{pmatrix}
0 & a_{12} & \cdots & a_{1\nu} \\
a_{21} & 0 & \cdots & a_{2\nu} \\
\vdots & \vdots & \ddots & \vdots \\
a_{k1} & a_{k2} & \cdots & 0 \\
\end{pmatrix} 
= A_{kh, \nu} \cdot \begin{pmatrix}
a_{11} & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{\nu1} & a_{\nu2} & \cdots & a_{\nu\nu} \\
\end{pmatrix} 
= M_{k\nu} 
\]

\[ (kH)^x (kH) \quad \text{induct.} \]

\[ = A_{kh, \nu} (a_{11} \ldots a_{\nu \nu}) \]

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Show that \( \det(A) = \begin{vmatrix}
a_{11} & \cdots & 0 \\
0 & a_{22} & \cdots \\
\vdots & \vdots & \ddots \\
0 & \cdots & a_{\nu \nu} \\
\end{vmatrix} = a_{11} \ldots a_{\nu \nu} \)

By induction

It's true for matrices \( A_{2 \times 2} \) (\( \nu = 2 \) for a \( A_{\nu \times 1} \)) \( A = [a_{ij}] \)

\[ \det(A) = a_{11} \]

Since

\[ a_{ij} a_{jk} = a_{11} a_{22} + 0 a_{21} \]

Assume true for \( n = \nu \), so

\[ \begin{vmatrix}
a_{11} & 0 & \cdots & 0 \\
0 & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & a_{\nu \nu} \\
\end{vmatrix} = a_{11} \ldots a_{\nu \nu} \]

Then for \( n = \nu + 1 \),

\[ \begin{vmatrix}
a_{11} & \cdots & 0 \\
a_{21} & a_{22} & \cdots \\
\vdots & \vdots & \ddots \\
0 & \cdots & a_{\nu \nu} \\
\end{vmatrix} 
= A(-1) \begin{vmatrix}
a_{11} & 0 & \cdots & 0 \\
0 & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & a_{\nu \nu} \\
\end{vmatrix} 
= A_{k+1, k+1} \]

\[ = A_{k+1, k+1} (a_{11} \ldots a_{\nu \nu}) \]

\[ = a_{k+1, k+1} \ldots a_{\nu \nu} \]