3.2 Properties of determinants.

We want to know if

a) $\det (A+B) = \det (A) + \det (B)$  
b) $\det (AB) = \det (A) \det (B)$  
c) $\det (kA) = k \det (A)$

for $A, B$ $n \times n$ matrices and $k$ scalar.

Before studying general results, let's consider some particular cases:

1) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 5 & -6 \end{bmatrix}$

$\Rightarrow A+B = \begin{bmatrix} 3 & 0 \\ 7 & -2 \end{bmatrix}$

and $\det (A) = -2$, $\det (B) = 3$, $\det (A+B) = -6$

Therefore $\det (A+B) = -6 = \det (A) + \det (B)$

2) However, for $A_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$, and

$C = \begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix}$, we have

$\det (A_2) = -2$, $\det (B_2) = 8$, $\det (C) = 6$

Thus $\det (C) = \det (A_2) + \det (B_2)$. 
3) Also,

\[
AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 24 & -30 \\ 17 & -31 \end{bmatrix}
\]

\[
det(A) = -2, \quad det(B) = 3, \quad det(AB) = -510 + 504 = -6
\]

Therefore,

\[
det(AB) = -6 = (-2) \times (3) = det(A) \times det(B).
\]

4) Consider

\[
3A = \begin{bmatrix} 3 & 9 \\ 6 & 12 \end{bmatrix}
\]

Then

\[
det(3A) = 36 - 54 = -18 \neq 3 \times (-2) = 3 \times det(A)
\]

In fact,

\[
det(3A) = -18 = 9 \times (-2) = 3^2 \times det(A).
\]

5) If \( D = \begin{bmatrix} 5 \times 1 & 5 \times 3 \\ 2 & 4 \end{bmatrix} \)

\[
\Rightarrow det(D) = 5 (1 \times 4) - 5 (3 \times 2) = 5 \times (4-6) =
\]

\[
= 5 \times (-2) = 5 \times det(A).
\]

Does it mean \( det \left( \begin{bmatrix} k \times a_{11} & k \times a_{12} & \cdots & k \times a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \right) = k \times det(A) \) ?
In this section, we also want to generalize Theorem 2 of Section 8.2, it means:

\[ A_{2 \times 2} \text{ is invertible if and only if } \det A \neq 0, \]

to general \( n \times n \) matrices.

In order to do this, we need to learn how to compute \( \det A \) from the determinants of the elementary matrices used in the row reduction of a matrix \( A \) to an echelon form.
Relationship between $\det A$ and $\det B$, where $B$ is obtained from $A$ by row reduction.

**Exercise**

\[
A = \begin{bmatrix}
1 & -2 & 3 & 1 \\
5 & -9 & 6 & 3 \\
-1 & 2 & -6 & -2 \\
2 & 8 & 6 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & -2 & 3 & 1 \\
0 & 1 & -9 & -2 \\
0 & 0 & -3 & -1 \\
0 & 12 & 0 & -1
\end{bmatrix}
\]

1 replace.

\[
\sim \begin{bmatrix}
1 & -2 & 3 & 1 \\
0 & 1 & -9 & -2 \\
0 & 0 & -3 & -1 \\
0 & 0 & 108 & 23
\end{bmatrix} \sim \begin{bmatrix}
1 & -2 & 3 & 1 \\
0 & 1 & -9 & -2 \\
0 & 0 & -3 & -1 \\
0 & 0 & 0 & -13
\end{bmatrix}
\]

How are $\det(A)$ and $\det(B)$ related?

$\det(B) = 1 \cdot 1 \cdot (-3) \cdot (-13) = 39$ \text{ thm 2}

$\det(A) = \det(B)$ \text{ thm 3 (Next page)}

**Observation:** In this particular example, the only row operations are "replacements." If row interchanges or scaling were performed then it would not be longer true that $\det A = \det B$ in general.
Thm 3: A is an $m \times n$ matrix

a) If $B$ is the matrix that results when a row or a column is multiplied by $K$.

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} ka_{11} & \ldots & ka_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

then, $\det(B) = K \det(A)$. 

b) If $B$ is obtained from $A$ by interchanging two rows or two columns, then $\det(B) = -\det(A)$. 

c) If $B$ results by adding a multiple of one row to another row of $A$ (similar for columns), then $\det(B) = \det(A)$. 

(look at book's table for $3 \times 3$ matrices).
### Example 2: Theorem 3 Applied to $3 \times 3$ Determinants

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ka_{11} \ a_{12} \ k\ a_{13} \ \ a_{21} \ a_{22} \ a_{23} \ \ a_{31} \ a_{32} \ a_{33}$</td>
<td>The first row of $A$ is multiplied by $k$</td>
</tr>
<tr>
<td>$</td>
<td>\ a_{21} \ a_{22} \ a_{23} \ \ a_{11} \ a_{12} \ a_{13} \ \ a_{31} \ a_{32} \ a_{33}$</td>
</tr>
<tr>
<td>$a_{21} \ a_{22} \ a_{23}$</td>
<td>The first and second rows of $A$ are interchanged.</td>
</tr>
<tr>
<td>$a_{11} \ a_{12} \ a_{13}$</td>
<td></td>
</tr>
<tr>
<td>$a_{31} \ a_{32} \ a_{33}$</td>
<td>$det(B) = -det(A)$</td>
</tr>
<tr>
<td>$a_{31} \ a_{32} \ a_{33}$</td>
<td>$</td>
</tr>
<tr>
<td>$a_{31} \ a_{32} \ a_{33}$</td>
<td>$det(B) = det(A)$</td>
</tr>
</tbody>
</table>

We will verify the equation in the last row of the table and leave the first two for the reader. With the help of Example 7 in Section 2.1 we obtain

$$
det(B) = (a_{11} + ka_{21})a_{22}a_{33} + (a_{12} + ka_{22})a_{23}a_{31} + (a_{13} + ka_{23})a_{23}a_{31}$$

$$
- a_{31}a_{22}(a_{13} + ka_{23}) - a_{33}a_{21}(a_{12} + ka_{22}) - a_{32}a_{23}(a_{11} + ka_{21})
$$

$$
det(A) + k(a_{21}a_{22}a_{33} + a_{22}a_{23}a_{31} + a_{23}a_{21}a_{32})
$$

$$
- a_{31}a_{22}a_{23} - a_{33}a_{21}a_{23} - a_{32}a_{23}a_{21}
$$

$$
det(A) + 0 = det(A)
$$

**Remark:** As illustrated by the first equation in Example 2, part (a) of Theorem 2.2.3 allows us to bring a "common factor" from any row (or column) through the determinant sign.

**Elementary Matrices** Recall that an elementary matrix results from performing a single elementary row operation on an identity matrix; thus, if we let $A = I_n$ in Theorem 2.2.3 so that we have $det(A) = det(I_n) = 1$, then the matrix $B$ is an elementary matrix, and the theorem yields the following result about determinants of elementary matrices.

**Theorem**

Let $E$ be an $n \times n$ elementary matrix.

(a) If $E$ results from multiplying a row of $I_n$ by $k$, then $det(E) = k$.

(b) If $E$ results from interchanging two rows of $I_n$, then $det(E) = -1$.

(c) If $E$ results from adding a multiple of one row of $I_n$ to another, then $det(E) = 1$. 
Proof of thm 3 part (a)

Let \( B = \begin{bmatrix}
  ka_1 & ka_2 & \ldots & ka_n \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix} \)

Applying Cofactor expansion along first row to compute \( \det B \) leads to

\[
\det B = ka_{11}c_{11} + ka_{12}c_{12} + \ldots + ka_{1n}c_{1n}
\]

\[
= k \left( a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n} \right)
\]

\[
= k \det A.
\]
Proof of theorem 3 part (b) for a 3x3 matrix $A$.

For a 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove the following: $B$ is obtained by row interchange

$$\det A = -\det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} = -\det B$$

Proof:

Renaming the entries of $B$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad \text{where} \quad b_{11} = a_{11}, \quad b_{12} = a_{22}, \quad b_{13} = a_{23}$$

$$b_{21} = a_{11}, \quad b_{22} = a_{12}, \quad b_{23} = a_{13}$$

$$b_{31} = a_{31}, \quad b_{32} = a_{32}, \quad b_{33} = a_{33}$$

Then

$$\det B = b_{11} b_{22} b_{33} + b_{21} b_{32} b_{13} + b_{31} b_{12} b_{23} - (b_{13} b_{22} b_{31} + b_{12} b_{21} b_{33} + b_{23} b_{32} b_{11})$$

$$= a_{21} a_{12} a_{33} + a_{11} a_{32} a_{23} + a_{22} a_{13} a_{31}$$

$$- (a_{23} a_{12} a_{31} + a_{22} a_{11} a_{33} + a_{13} a_{32} a_{21}) =$$

$$= - \left( a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{12} a_{23} a_{31} - (a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{23} a_{32} a_{11}) - (a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{23} a_{32} a_{11}) \right)$$

$$= -\det A.$$
Exercises:

1) \[ A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}, \quad \det A = ? \]

\[
\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -5 \end{vmatrix} = (-3)(-55) = 165.
\]

2) \[
\begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = (-1)(-1)(33) = (1)(-18) = -18.
\]
Lemmat: If a $B_{mn}$ matrix is such that two of its rows are identical then
\[ \det B = 0. \]

Proof: It is done for a $3 \times 3$ matrix. For the general case similar ideas can be applied.

Let
\[ B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

Then using cofactor expansion along first row
\[ \det B = a_{21} \begin{vmatrix} a_{32} & a_{33} \\ a_{31} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \]
\[ = a_{21} a_{33} a_{32} - a_{21} a_{32} a_{33} - a_{22} a_{31} a_{33} + a_{23} a_{31} a_{32} + a_{23} a_{32} a_{31} - a_{23} a_{31} a_{32} \]
\[ = 0 \]

Thus, \[ \boxed{\det B = 0} \]
Lemma 2. Let

\[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \]

Then

\[ \det \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \det A + \det C. \]

Proof: Applying Cofactor expansion along the first row of \(D\) to compute its determinant leads to

\[ \det D = (a_{11}+b_{11})C_{11} + (a_{12}+b_{12})C_{12} + \cdots + (a_{1n}+b_{1n})C_{1n} \]

\[ = (a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}) + (b_{11}C_{11} + b_{12}C_{12} + \cdots + b_{1n}C_{1n}) \]

\[ = \det A + \det C \]
Proof of theorem 3 part (c).

Let
\[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} a_{11} + ka_{11} & \cdots & a_{1n} + ka_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \]

Then, using lemma 2
\[ \det B = \det A + \det \begin{bmatrix} ka_{11} & \cdots & ka_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \]

\[ \text{part (a) Thm 3} \]
\[ = \det A + k \det \begin{bmatrix} a_{21} & \cdots & a_{2n} \\ a_{21} & \cdots & a_{2n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \]
\[ \text{lemma 1} \]
\[ = \det A + k \cdot 0 = \det A. \]

Therefore, \( \det A \) does not change if a multiple of a row is added to another row.
Assume $A$ has been reduced to an echelon form $U$ by row replacements and row interchanges.

Then, from Thm 3

$$\det A = (-1)^r \det U$$

Since $U$ is in echelon form and $U_{mn}$

$$\det U = U_{11} \ldots U_{nn} \quad \text{(Triangular matrix)}$$

Then

$$\det A = \begin{cases} 
(-1)^r \left( \text{product of pivots} \right), & \text{if invertible} \\
0, & \text{if not invertible}
\end{cases}$$
Now, we want to establish a relationship between invertibility of matrices and their determinants.

First, we will learn how to compute $\det A$ from determinants of elementary matrices.

**Corollary 1 from thm 3.**

Let $E$ be an $n \times n$ elementary matrix

a) If $E$ results from multiplying a row of $I_n$ by $k$, then $\det E = k$.

b) If $E$ results from interchanging two rows of $I_n$, then $\det E = -1$.

c) If $E$ results from adding a multiple of one row of $I_n$ to another row, then $\det E = 1$.

**Proof:** It is an immediate consequence of thm 3 and the fact that $\det I_n = 1$. 
Corollary 2 from thm 3

If $B_{nxn}$ matrix and $E_{nxn}$ elementary matrix, then
\[
\det EB = \det E \det B.
\]

Proof: There are three cases depending on $E$.

Case 1: $E = \begin{bmatrix} 1 & 0 \\ 0 & k \\ \vdots & \vdots \\ 0 & 0 \\ & & 1 \end{bmatrix}$, \( \det E = \frac{k}{k} \)

Then $EB = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & k b_{22} & \ldots & k b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{bmatrix}$

And $\det EB \overset{\text{thm 3}}{=} K \det B = \det E \det B$

Case 2: Obtained from

If $E$ is interchanging two rows of $In$

Then $\det E = (-1)^{ij}$

And $EB$ is the matrix that results from interchanging the same rows of $B$. \( \text{Thm 3} \)

Therefore, $\det EB = -\det B = \det E \det B$. 
Case 3.-

If $E$ is obtained by adding a multiple of one row of $I_n$ to another row then

$$\det E = 1$$

and $EB$ is the matrix that results from adding a multiple of the same row of $B$ to the same another row of $B$. Therefore,

$$\det EB = \det B = \det E \det B.$$  

Corollary 3.-

If $B_{n \times n}$ matrix and $E_1, E_2, \ldots, E_r$ are $n \times n$ elementary matrices,

then

$$\det (E_1 E_2 E_3 \ldots E_r B) = \det E_1 \det E_2 \ldots \det E_r \det B.$$  

Proof.-

$$\det (E_1 E_2 E_r B) = \det E_1 \det (E_2 E_3 \ldots E_r B)$$  

Corollary 2.

$$\det E_1 \det E_2 \det (E_3 \ldots E_r B) = \ldots = \det E_1 \det E_2 \ldots \det E_r \det B$$
**Thm 4** A $m \times n$ matrix is invertible if and only if $\det A \neq 0$.

**Proof**

$(\rightarrow)$ A matrix is invertible implies that

$$A = E_1 E_2 \ldots E_r I_n$$

(Inv. matrix thm).

Then using Corollary 3

$$\det A = \det E_1 \det E_2 \ldots \det E_r \det I_n \neq 0$$

$(\leftarrow)$ Consider $R$ the row-reduced echelon form of $A$.

then $R = I_n$ or $R$ has a row of zeros (Key Lemma)

Also

$$R = E_1 E_2 \ldots E_r A$$

Therefore,

$$\det R = \det E_1 \det E_2 \ldots \det E_r \det A \neq 0$$

Then, $R$ cannot have a row of zeros, otherwise

$$\det R = 0$$

Therefore, the only possibility for $R$ is to be $I_n$

i.e., $R = I_n$.

And that means $A \sim R = I$ is invertible according to Inv. matrix thm.
**Thm 6.** \( A_{nn} \) and \( B_{nn} \)

\[
\det AB = \det A \det B
\]

**Proof.**

Two cases arise depending on \( A \).

**Case 1.** \( A \) is singular \( \Rightarrow \) \( AB \) is singular. (Statement in page 6, Sect. 2.3)

\[
\Rightarrow \det AB = 0 = 0, \det B = \det A \det B
\]

**Case 2.** \( A \) is invertible \( \Rightarrow \) \( A \sim I_n \)

\[
\Rightarrow E_r \ldots E_1 A = I_n \Rightarrow A = E_r^{-1} \ldots E_1^{-1} \quad \text{also elem matrices}
\]

\[
\Rightarrow \det AB = \det \left( E_r^{-1} \ldots E_1^{-1} B \right) = \det E_r^{-1} \ldots \det E_1^{-1} \det B
\]

**Corollary 3**

\[
\det \left( E_1^{-1} E_2^{-1} \ldots E_r^{-1} \right) \det B = \det A \det B
\]
We can easily prove that
\[
\det(A) = -A_{21}M_{21} + A_{22}M_{22} - A_{23}M_{23} \quad \text{ (elimination along 2nd row)}
\]
\[
\det(A) = A_{31}M_{31} - M_{32} + A_{33}M_{33} \quad \text{ (along 3rd row)}
\]
\[
\det(A) = -A_{12}M_{12} + A_{22}M_{22} - A_{32}M_{32} \quad \text{ (along 2nd col.)}
\]
\[
\det(A) = A_{13}M_{13} - A_{23}M_{23} + A_{33}M_{33} \quad \text{ (along 3rd col.)}
\]

However,
\[
A_{11}M_{31} - A_{12}M_{32} + A_{13}M_{33} = \left(\begin{array}{c}
\text{entries and M's correspond to different rows}\n\end{array}\right)
\]
\[
= 1(3) - 4(0) - 3(1) = 0
\]

This is a particular case of a general result.

**Definition:** The **determinant** $M_{ij}$ of the submatrix that remains after the $i$th row and $j$th column are deleted from $A$ is called the **minor of entry** $A_{ij}$.

The number $\text{C}_{ij} = (-1)^{i+j}M_{ij}$ is called the **cofactor** of entry $a_{ij}$. 
In our previous example, we showed a different way to obtain the $\det(A)$ for a $3 \times 3$ matrix. This result can be extended to any matrix $n \times n$.

**Thus 2.4.1 Expansion by cofactors.**

\[
\det(A) = a_{ij} C_{ij} + a_{2j} C_{2j} + \ldots + a_{nj} C_{nj}
\]

(Expansion along $j$th column)

\[
\det(A) = a_{1i} C_{1i} + a_{2i} C_{2i} + \ldots + a_{ni} C_{ni}
\]

(cofactor exp. along $i$th row)

$1 \leq i \leq n, \quad 1 \leq j \leq n.$

**Efficient method to evaluate determinants**

Cofactor expansion + row or column operations.

**Exercise #9.**

Find $\det(A)$

\[
\det(A) = \begin{vmatrix}
3 & 3 & 0 & 5 \\
2 & 2 & 0 & -2 \\
4 & 1 & -3 & 0 \\
2 & 10 & 3 & 2
\end{vmatrix} = -3 \begin{vmatrix}
3 & 5 \\
2 & -2
\end{vmatrix} = -3 \begin{vmatrix}
2 & 2 \\
6 & 10
\end{vmatrix} = -3 \begin{vmatrix}
6 & 2 \\
11 & 2
\end{vmatrix} = -3(50 + 30) = -240
\]