4.3 Linearly Independent Sets, Bases.

Consider the following vectors:

**Example 1. Vector space \( \mathbb{R}^2 \)**
\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}
\]

**Example 2. Vector space \( \mathbb{P}_2 \)**
\[
\vec{p}_1 = 3 + x + x^2, \quad \vec{p}_2 = 2 - x + 5x^2, \quad \vec{p}_3 = 4 - 3x^2.
\]

For the set of vectors in Example 1, we know how to determine if they are linearly independent or not. For this, we consider the linear combination
\[
C_1 \vec{v}_1 + C_2 \vec{v}_2 + C_3 \vec{v}_3 + C_4 \vec{v}_4 \text{ and equal to } 0
\]
\[
C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + C_4 \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1.1)
\]

Then, we determine the values of the weights or coefficients \(C_1, C_2, C_3, C_4\). If they are all zeros then the set \(S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}\) is linearly independent. Otherwise, the set \(S\) is linearly dependent.
We can proceed with equation (1.1) to determine $c_1, c_4$. In fact, (1.1) leads to
\[
\begin{bmatrix}
1 & 2 & 4 & 0 \\
0 & 1 & 3 & -2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
This is the same matrix equation that we studied in Section 4.2. Its augmented matrix can be row-reduced as
\[
\begin{bmatrix}
1 & 2 & 4 & 0 & 0 \\
0 & 1 & 3 & -2 & 0
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 0 & -2 & 4 & 0 \\
0 & 1 & 3 & -2 & 0
\end{bmatrix}
\]
Therefore,
\[
\begin{cases}
  c_1 = 2c_3 - 4c_4 \\
  c_2 = -3c_3 + 2c_4 \\
  c_3, c_4 \text{ are free variables}
\end{cases}
\quad (2.1)
\]
It means $c_3$ and $c_4$ can take any real value.
As a consequence, the set $S$ is linearly dependent.
This was expected, because we know already that two vectors in $\mathbb{R}^2$ not in the same line span $\mathbb{R}^2$.
In this case, we have two more vectors than needed to span $\mathbb{R}^2$. 

The concepts of linear independence and linear dependence can easily be extended to general vector spaces such as the space of polynomials $P_2$ of degree $\leq 2$.

**Def.** The set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of a general vector space $V$ are linearly independent if

$$C_1 \vec{v}_1 + C_2 \vec{v}_2 + \ldots + C_p \vec{v}_p = \vec{0}$$

implies $C_1 = 0, \ldots, C_p = 0$.

Otherwise, if any of the coefficients $C_1, \ldots, C_p$ is nonzero, the set $S$ is linearly dependent.

---

So, to determine that the set of polynomials of degree $\leq 2$ given in example 2 is linearly independent or not, we consider the linear combination:

$$C_1 \vec{p}_1 + C_2 \vec{p}_2 + C_3 \vec{p}_3 = \vec{0}$$

or

$$C_1 (3 + x + x^2) + C_2 (2 - x + 5x^2) + C_3 (4 - 3x^2) = \vec{0}$$

$$= 0 + 0x + 0x^2.$$
\[ 3c_1 + 2c_2 + 4c_3 + (c_1 - c_2)x + (c_1 + 5c_2 - 3c_3)x^2 = 0 + 0x + 0x^2. \]

Which leads to the linear system of equations (\(^\ast\))

\[
\begin{align*}
3c_1 + 2c_2 + 4c_3 &= 0 \\
3c_1 - c_2 &= 0 \\
4c_1 + 5c_2 - 3c_3 &= 0
\end{align*}
\]

In matrix form:
\[
\begin{bmatrix}
3 & 2 & 4 \\
1 & -1 & 0 \\
1 & 5 & -3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Augmented matrix
\[
\begin{bmatrix}
3 & 2 & 4 & 0 \\
1 & -1 & 0 & 0 \\
1 & 5 & -3 & 0
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \tag{4.1}
\]

Therefore, \( c_1 = 0, \ c_2 = 0, \ c_3 = 0 \)

And the set \( S = \{ \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \} = \)
\[
\{3 + x + x^2, 2 - x + 5x^2, 4 - 3x^2\}
\]
is linearly independent.

\(^\ast\) According to the fundamental thm of algebra, a polynomial of degree 2 cannot have more than two roots unless it is the zero polynomial.
Remarks: These statements are immediate consequences of definition.

i) If $\vec{v} \in V$ is such that $\vec{v} \neq \vec{0}$, then the set $S = \{\vec{v}\}$ is linearly independent.

ii) A set $S = \{\vec{v}_1, \vec{v}_2\} \subset V$ is linearly dependent if and only if one of the two vectors is multiple of the other.

iii) If $\vec{0} \in S$, then $S$ is linearly dependent.

Theorem 4. $V$ is a vector space

A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$, with $\vec{v}_1 \neq \vec{0}$ is linearly dependent if and only if there exists $\vec{v}_j$ (j > 1) such that $\vec{v}_j = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_{j-1} \vec{v}_{j-1}$.

Proof: $(\Rightarrow)$ If $S$ is linearly dependent, then there exists a linear combination

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_{p-1} \vec{v}_{p-1} + c_p C_{j+p} = \vec{0}$$

(5.1)

Where not all the coefficients are zero.

Let $j$ be the largest subscript for which $C_j \neq 0$. 
Obviously, if not \( c_j \vec{v}_j = 0 \), which is impossible because \( c_j \neq 0 \) and \( \vec{v}_j \neq 0 \).

Therefore, solving for \( \vec{v}_j \) in equation (5.1) results

\[
\vec{v}_j = -\frac{c_1}{c_j} \vec{v}_1 - \frac{c_2}{c_j} \vec{v}_2 - \cdots - \frac{c_{j-1}}{c_j} \vec{v}_{j-1}
\]

and \( \vec{v}_j \) is a linear combination of its preceding vectors.

\((\leftarrow)\) If \( \vec{v}_j = c_1 \vec{v}_1 + \cdots + c_{j-1} \vec{v}_{j-1} \)

then \( c_1 \vec{v}_1 + \cdots + c_{j-1} \vec{v}_{j-1} - \vec{v}_j = \vec{0} \)

As a consequence, \( S \) is linearly dependent.

The important concept in this section is the concept of a basis for a vector space \( V \).

**Def.** \( H \subset V \) is a Subspace of \( V \).

The Set \( B = \{ b_1, b_2, \ldots, b_p \} \) is a basis for \( H \) if

i) \( B \) is a linearly independent set.

ii) \( \text{Span}(B) = H \).
Examples:

i) The set \( B = \{ \hat{e}_1, \hat{e}_2 \} = \{ [1], [0] \} \subset \mathbb{R}^2 \) is a basis for \( \mathbb{R}^2 \). They are clearly linearly independent and they span \( \mathbb{R}^2 \), because any vector \( \hat{x} = \begin{bmatrix} x \end{bmatrix} \in \mathbb{R}^2 \) can be written as \( \hat{x} = x \hat{e}_1 + y \hat{e}_2 \).

So any \( \hat{x} \in \mathbb{R}^2 \) is a linear combination of \( \hat{e}_1 \) and \( \hat{e}_2 \).

ii) Similarly,

\[ B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^n. \]

This basis is called the standard basis for \( \mathbb{R}^n \).

iii) Consider \( P_n \) (polynomials of degree \( \leq n \)).

\[ B = \{ 1, x, x^2, \ldots, x^n \} \subset P_n \text{ is a basis for } P_n. \]

\( P_n \) is a standard basis. It spans \( P_n \), because any polynomial in \( P_n \) can be written as

\[ p(x) = a_0 + a_1 x + \cdots + a_n x^n. \]

It's also linearly independent, because the only possibility for

\[ a_0 + a_1 x + \cdots + a_n x^n = 0 \]

is \( \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = 0 \) and \( a_i = 0 \).
Remarks:

i) Any set of $n$ linearly independent vectors in $\mathbb{R}^n$

\[ B = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \text{ in } \mathbb{R}^n \] is a basis for $\mathbb{R}^n$.

We only need to prove that $B$ spans $\mathbb{R}^n$.

But, this is trivially true from the invertible matrix theorem because if we define the matrix $A$ as

\[ A = [\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_n] \]

then any $\vec{b} \in \mathbb{R}^n$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ because the linear system

\[ A \vec{x} = \vec{b}, \quad \text{always has a solution.} \]

ii) The set of polynomials

\[ B = \{ \bar{p}_1, \bar{p}_2, \bar{p}_3 \} = \{ 3 + x + x^2, 2 - x + 5x^2, 4 - 3x^2 \} \text{ in } \mathbb{R}_2 \]

is a basis for $\mathbb{R}_2$.

We already proved that $B$ is linearly independent.

Prove that $\text{Span } (B) = \mathbb{R}_2$. 

Show
\[ \text{Span } \langle B \rangle = \mathbb{R}_2 \]

Consider an arbitrary polynomial \( \tilde{q} \in \mathbb{R}_2 \)
\[ \tilde{q} = d_0 + d_1 x + d_2 x^2, \quad d_0, d_1, d_2 \in \mathbb{R}. \]

Want to prove \( \tilde{q} \in \text{Span } \langle B \rangle \)
Equivalent to find \( c_1, c_2 \) and \( c_3 \) such that
\[ p_1 (3 + x + x^2) + p_2 (x - x + 5x^2) + p_3 (4 - 3x^2) \]
\[ = d_0 + d_1 x + d_2 x^2 \]

Which leads to
\[
\begin{cases}
3c_1 + 2c_2 + 4c_3 = d_0 \\
-c_1 - c_2 = d_1 \\
c_1 + 5c_2 - 3c_3 = d_2
\end{cases}
\]
or in matrix form:
\[
\begin{bmatrix}
3 & 2 & 4 \\
-1 & -1 & 0 \\
1 & 5 & -3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= \begin{bmatrix}
d_0 \\
d_1 \\
d_2
\end{bmatrix}
\]

Since \( A = I_3 \) (page 4 Eq. (4.11))
\( A \) is invertible and the matrix equation \( A\hat{c} = \hat{d} \) has
a solution. Therefore, \( \tilde{q} \in \text{Span } \langle B \rangle \). Moreover, this solution
is unique.
Spanning Set theorem

Thm 5. \[ S = \{ \vec{v}_1, \ldots, \vec{v}_p \} \subset \mathbb{V} \text{ (vector space)} \]

\[ H = \text{Span} \{ \vec{v}_1, \ldots, \vec{v}_p \} \]

then

a) If \( \vec{v}_k \in S \) is a linear combination of the others, i.e.,

\[ \vec{v}_k = c_1 \vec{v}_1 + \cdots + c_{k-1} \vec{v}_{k-1} + c_{k+1} \vec{v}_{k+1} + \cdots + c_p \vec{v}_p \]

then \( \text{Span} \{ \vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{v}_{k+1}, \ldots, \vec{v}_p \} = H \).

b) If \( H \neq \{ \vec{0} \} \) then there is \( \mathcal{B} \subset \mathcal{H} \) such that \( \mathcal{B} \) is a basis for \( H \).

Proof: a) We want to prove that any \( \vec{x} \in H \)

Can be written as

\[ \vec{x} = d_1 \vec{v}_1 + d_2 \vec{v}_2 + \cdots + d_{k-1} \vec{v}_{k-1} + d_{k+1} \vec{v}_{k+1} + \cdots + d_p \vec{v}_p \]

(9.1)

In fact, \( \vec{x} \in H \) if and only if

\[ \vec{x} = q_1 \vec{v}_1 + \cdots + q_{k-1} \vec{v}_{k-1} + q_k \vec{v}_k + q_{k+1} \vec{v}_{k+1} + \cdots + q_p \vec{v}_p \]
Or
\[ \begin{align*}
\dot{x} &= a_0 \dot{v}_1 + \cdots + a_{k-1} \dot{v}_{k-1} + a_k (c_1 \dot{v}_1 + \cdots + c_{k-1} \dot{v}_{k-1} + c_k \dot{v}_k + \cdots + c_p \dot{v}_p) \\
+ a_{k+1} \dot{v}_{k+1} + \cdots + a_p \dot{v}_p = \\
= (a_1 + a_k c_1) \dot{v}_1 + \cdots + (a_{k-1} + a_k c_{k-1}) \dot{v}_{k-1} + (a_{k+1} + a_k c_k) \dot{v}_k + \cdots \\
+ \cdots + (a_{p-1} + a_k c_p) \dot{v}_p
\end{align*} \]

by renaming \( a_i + a_k c_i = a_i \), \( i = 1, \ldots, k, k+1, \ldots, p \).

We obtain (9.1), and \( \dot{x} \in \text{Span} \{ \dot{v}_1, \dot{v}_2, \ldots, \dot{v}_{k-1}, \dot{v}_{k+1}, \ldots, \dot{v}_p \} \).

b) If the spanning set of \( H, S = \{ \dot{v}_1, \ldots, \dot{v}_p \} \) is linearly independent then \( S \) is a basis for \( H \).

Otherwise, we use the procedure outlined in part (a) of this theorem, and remove vectors successively until the new set becomes linearly independent. This is possible because \( H \neq \{0\} \) implies that at least one of \( \dot{v}_j \), \( j = 1, \ldots, p \).

Apply this procedure to our Example 1 and show that \( B = \{ [1], [i] \} \) is a basis for \( \text{Col}(A) \)

where \( A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix} \).
Finding a basis for Null(A).

In Section 4.2, we found the null space for the matrix

\[ B = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix} \]

To do this, we considered the homogeneous linear system \( B \vec{x} = \vec{0} \)

row-reduced the augmented matrix and found out that any vector \( \vec{x} \in \text{Null}(B) \) is a linear combination of the following vectors in \( \mathbb{R}^4 \)

\[ \vec{x} = c_1 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}. \]

It means

\[ \text{Null}(B) = \text{Span} \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \]

These two vectors form a linearly independent set, because they are not multiples of each other. Therefore, the set

\[ \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \]

forms a basis for \( \text{Null}(B) \).
This procedure followed to find \( \text{Null}(B) \) always leads to a basis for \( \text{Null}(B) \) for any matrix \( B \).

Finding a basis for \( \text{Col}(A) \)

Consider again
\[
B = \begin{bmatrix}
1 & 2 & 4 & 0 \\
0 & 1 & 3 & -2
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -2 & 4 \\
0 & 1 & 3 & -2
\end{bmatrix}
= \begin{bmatrix}
b_1 & b_2 & b_3 & b_4
\end{bmatrix}
\]

Clearly,
\[
\begin{align*}
b_3 &= -2b_1 + 3b_2 \\
b_4 &= 4b_1 - 2b_2
\end{align*}
\]

Moreover, the set \( S = \{ \ddot{d}_1, \ddot{d}_2 \} = \{ [6], [2] \} \) is linearly independent.

An important observation here is that the same linear dependence relationship occurs among the column vectors of \( \ddot{B} \) among the column vectors of \( B \). It means
\[
\begin{align*}
b_3 &= -2\ddot{b}_1 + 3\ddot{b}_2 \\
b_4 &= 4\ddot{b}_1 - 2\ddot{b}_2
\end{align*}
\]

and also the set \( R = \{ \ddot{b}_1, \ddot{b}_2 \} = \{ [6], [2] \} \)

is linearly independent.

Why is this true in general?
Therefore,

\[ \text{Col}(B) = \text{Span} \{ \tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4 \} \]

Spanning set thm.

\[ \text{Span} \{ \tilde{b}_1, \tilde{b}_2 \} \]

and \( R = \{ \tilde{b}_1, \tilde{b}_2 \} \) is linearly indep.

Therefore, \( R \) is a basis for \( \text{Col}(B) \).

The procedure applied in the previous example can be applied to any matrix \( A \) to find a basis for \( \text{Col}(A) \). This procedure is supported by the following thm.

\textbf{Thm 6} The pivot columns of a matrix \( A \) form a basis for \( \text{Col}(A) \).

Before proving thm 6, we will prove some lemmas.
Important observations about $\text{Nul}(B)$ and $\text{col}(B)$

\[
B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}, \quad B \in \mathbb{R}^{2 \times 4}
\]

\[\bar{x} \in \text{Nul}(B) \iff B\bar{x} = 0\]

\[\bar{x} \in \text{col}(B) \iff \bar{x} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^2\]

Computation of $\text{Nul}(B)$ Based on the statement $B \sim \tilde{B}$ then $B\bar{x} = 0 \iff \tilde{B}\bar{x} = 0$.

\[
\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -2 \end{bmatrix} \Rightarrow x_1 = 2x_3 - 4x_4 \quad x_2 = -3x_3 + 2x_4 \quad x_3, x_4 \text{ free}
\]

Therefore,

\[\bar{x} \in \text{Nul}(B) \iff \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - 4x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad x_3, x_4 \text{ arbitrary real numbers}
\]

So

\[\text{Nul}(B) = \text{Span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^4\]

If we consider the linear transformation

\[T : \mathbb{R}^4 \rightarrow \mathbb{R}^2, \quad \bar{x} \mapsto B\bar{x}\]

We clearly see

\[\text{Domain } T = \mathbb{R}^4, \quad \text{Null}(B) \subset \text{Dom } T\]

\[\text{Range } T \subset \mathbb{R}^2, \quad \text{col}(B) = \text{Range } T\]
Relationship between $\text{Nul}(A)$ and $\text{Nul}(\tilde{A})$.

**Lemma 1.** Elementary row operations do not change the null space of a matrix $A$.

Equivalent to say:

If $A \sim \tilde{A}$ then $\text{Nul}(A) = \text{Nul}(\tilde{A})$

or $A\tilde{x} = \mathbf{0} \iff \tilde{A}\tilde{x} = \mathbf{0}$

**Proof.**

If $A \sim \tilde{A}$, then $\tilde{A} = E_r \ldots E_1 A$,

where $E_i$ $(i=1,\ldots,r)$ are elementary matrices.

Thus, if $\tilde{x} \in \text{Nul}(A) \Rightarrow A\tilde{x} = \mathbf{0} \Rightarrow \tilde{x} = (E_r \ldots E_1 A)\tilde{x} = (E_r E_{r-1} \ldots E_1)\tilde{A}\tilde{x} = \mathbf{0}$

$\Rightarrow \tilde{x} \in \text{Nul}(\tilde{A}) \Rightarrow \text{Nul}(A) \subseteq \text{Nul}(\tilde{A})$

Similarly,

if $\tilde{x} \in \text{Nul}(\tilde{A}) \Rightarrow \tilde{A}\tilde{x} = \mathbf{0} \Rightarrow$

$A\tilde{x} = (E_r \ldots E_1 \tilde{A})\tilde{x} = (E_r \ldots E_1)\tilde{A}\tilde{x} = \mathbf{0}$.

$\Rightarrow \tilde{x} \in \text{Nul}(A) \Rightarrow \text{Nul}(\tilde{A}) \subseteq \text{Nul}(A)$.

**Remark:** The practical value of this theorem is that to find $\text{Null}(A)$ for a given matrix $A$, first row-reduce $A$ to $\tilde{A}$ and then find $\text{Null}(\tilde{A})$ which is the same as $\text{Null}(A)$. 
Procedure to find a basis for $\text{Col}(A)$.

Consider the matrix $A_{4 \times 5}$

\[
A = \begin{bmatrix}
\hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 & \hat{a}_5 \\
1 & 2 & 3 & -4 & 8 \\
1 & 2 & 0 & 2 & 8 \\
2 & 4 & -3 & 10 & 9 \\
3 & 6 & 0 & 6 & 9
\end{bmatrix}
\sim \begin{bmatrix}
\hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \hat{b}_4 & \hat{b}_5 \\
1 & 2 & 0 & 2 & 5 \\
0 & 0 & 3 & -6 & 3 \\
0 & 0 & 0 & 0 & -7 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} = \tilde{A}
\]

Obviously, $\text{Col}(A) \neq \text{Col}(\tilde{A})$.

For instance, \[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \notin \text{Col}(\tilde{A}) \quad \text{why not?}
\]

However, \[\hat{b}_2 = 2\hat{b}_1, \quad \hat{a}_2 = 2\hat{a}_1\]
\[\hat{b}_4 = \hat{b}_2 - 2\hat{b}_3, \quad \hat{a}_4 = \hat{a}_2 - 2\hat{a}_3\]

Also, the set $\tilde{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\
0 \\ 0 \\ 0 \\ 0 
\end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\
3 \\ 0 \\ 0 \\ 0 
\end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 3 \\ 0 
\end{bmatrix} \right\}$ is lin indep.

Because none of its vectors is lin. comb. of the preceding ones.

Therefore, $\text{Span}(\tilde{B}) = \text{Col}(\tilde{A})$, and $\tilde{B}$ is a basis for $\text{Col}(\tilde{A})$.

It can be shown that the set $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \right\}$ is also lin indep.

And $\text{Span}(B) = \text{Col}(A)$, It means $B$ is a basis for $\text{Col}(A)$. 

Lemma 2. If \( A \sim \tilde{A} \) then

i) Columns of \( A \) have exactly the same linear dependence relationship as the columns of \( \tilde{A} \).

For instance,

a) A subset of column vectors of \( A \) is linearly independent if the corresponding column vectors of \( \tilde{A} \) form a set linearly independent.

ii) A subset of column vectors of \( A \) forms a basis for \( \text{Col}(A) \) if and only if the corresponding column vectors of \( \tilde{A} \) form a basis for \( \text{Col}(\tilde{A}) \).

Proof. From lemma 1, \( \hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) is a solution of

\[
A \hat{x} = \bar{0} \quad \text{or} \quad x_1 \tilde{a}_1 + x_2 \tilde{a}_2 + \ldots + x_n \tilde{a}_n = \bar{0} \quad (16.1)
\]

if and only if \( \hat{x} \) is a solution of

\[
\tilde{A} \tilde{x} = \bar{0} \quad \text{or} \quad x_1 \tilde{b}_1 + x_2 \tilde{b}_2 + \ldots + x_n \tilde{b}_n = \bar{0} \quad (16.2)
\]

Therefore, \( \{ \tilde{a}_1, \ldots, \tilde{a}_n \} \) is linearly independent if and only if \( \{ \tilde{b}_1, \ldots, \tilde{b}_n \} \) is also linearly independent.
Also, $\tilde{b}_j (j = 1, n)$ is linearly dependent on $\tilde{b}_1, \tilde{b}_{j-1}, \tilde{b}_{j+1}, \ldots, \tilde{b}_n$ if and only if $\tilde{a}_j$ is linearly dependent on $\tilde{a}_1, \tilde{a}_{j-1}, \tilde{a}_{j+1}, \ldots, \tilde{a}_n$

Moreover, the linear dependence relationship is identical in both cases, because the coefficients (or components of $\tilde{x}$) are exactly the same in (16.1) and (16.2).

ii) If a subset $B$ of the columns of $A$ forms a basis for column $A$, then $B$ is linearly independent, and the corresponding subset $\tilde{B}$ of the columns of $\tilde{A}$ is also linearly independent from part (i).

Also, $\text{Span} \ (B) = \text{Col} \ (A)$, then any linear combination including another vector of the columns of $A$ will be linearly dependent.

From (i), this same property is true for the columns of $\tilde{A}$. As a consequence, $\text{Span} \ (\tilde{B}) = \text{Col} \ (\tilde{A})$, and $\tilde{B}$ is a basis for $\tilde{A}$. (The reciprocal is also true.

If $B$ is a basis for $\text{Col}(A)$,

$\Rightarrow B$ is a basis for $\text{col}(A)$.

identical proof.
Proof of Theorem 6

"Pivot columns of a matrix \( A \) form a basis for \( \text{Col}(A) \)."

First, we consider the row-reduced matrix \( \tilde{A} \)

\[
\tilde{A} = \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 & \ldots & \tilde{b}_n \end{bmatrix}
\]

If \( \tilde{b}_j \) is the first pivot column, then the next pivot column \( \tilde{b}_k \) is linearly independent with \( \tilde{b}_j \) because it has its leading "1" in the row below to when \( \tilde{b}_j \) has its leading "1".

\[
\begin{bmatrix}
\tilde{b}_j & \tilde{b}_p \\
0 & \cdots & 0 \\
\vdots & 0 & \ddots & \vdots \\
0 & 1 & \cdots & 0
\end{bmatrix}
\]

The next pivot column will also be linearly independent with the preceding two columns. This process can be continued until there are no more pivot columns.

Thus, the set obtained \( \tilde{B} = \{ \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n \} \) is linearly independent.

Moreover, any column in between is linearly dependent on the previous ones.
Therefore,

\[ \text{Span } (\tilde{\mathbf{B}}) = \text{Col } (\tilde{\mathbf{A}}). \]

And \( \tilde{\mathbf{B}} \) is a basis for \( \text{Col } (\tilde{\mathbf{A}}) \).

From the previous lemma, the corresponding set \( \tilde{\mathbf{B}} \) of columns of \( \mathbf{A} \) is also a basis for \( \text{Col } (\mathbf{A}) \). This set \( \tilde{\mathbf{B}} \) is by definition formed by the pivot columns of matrix \( \mathbf{A} \), which is what we want to prove.