Observe that the zero vector is orthogonal to every vector in \( \mathbb{R}^n \) because \( \mathbf{0} \cdot \mathbf{v} = 0 \) for all \( \mathbf{v} \).

The next theorem provides a useful fact about orthogonal vectors. The proof follows immediately from the calculation in (1) above and the definition of orthogonality. The right triangle shown in Fig. 6 provides a visualization of the lengths that appear in the theorem.

**The Pythagorean Theorem**

Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal if and only if \( \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \).

**Orthogonal Complements**

To provide practice using inner products, we introduce a concept here that will be of use in Section 6.3 and elsewhere in the chapter. If a vector \( \mathbf{z} \) is orthogonal to every vector in a subspace \( W \) of \( \mathbb{R}^n \), then \( \mathbf{z} \) is said to be **orthogonal to** \( W \). The set of all vectors \( \mathbf{z} \) that are orthogonal to \( W \) is called the **orthogonal complement** of \( W \) and is denoted by \( W^\perp \) (and read as "\( W \) perpendicular" or simply "\( W \) perp").

**Example 6** Let \( W \) be a plane through the origin in \( \mathbb{R}^3 \), and let \( L \) be the line through the origin and perpendicular to \( W \). If \( \mathbf{z} \) and \( \mathbf{w} \) are nonzero, \( \mathbf{z} \) is on \( L \), and \( \mathbf{w} \) is in \( W \), then the line segment from \( \mathbf{0} \) to \( \mathbf{z} \) is perpendicular to the line segment from \( \mathbf{0} \) to \( \mathbf{w} \); that is, \( \mathbf{z} \cdot \mathbf{w} = 0 \). See Fig. 7. So each vector on \( L \) is orthogonal to every \( \mathbf{w} \) in \( W \). In fact, \( L \) consists of all vectors that are orthogonal to the \( \mathbf{w}'s \) in \( W \), and \( W \) consists of all vectors orthogonal to the \( \mathbf{z}'s \) in \( L \). That is,

\[
L = W^\perp \quad \text{and} \quad W = L^\perp
\]

The following two facts about \( W^\perp \), with \( W \) a subspace of \( \mathbb{R}^n \), are needed later in the chapter. Proofs are suggested in Exercises 29 and 30. Exercises 27–31 provide excellent practice using properties of the inner product.

1. A vector \( \mathbf{x} \) is in \( W^\perp \) if and only if \( \mathbf{x} \) is orthogonal to every vector in a set that spans \( W \).
2. \( W^\perp \) is a subspace of \( \mathbb{R}^n \).