Chap I  Systems of Linear Equations and Matrices

Consider the following equations:

a) $2x - y = 1$

b) $y - x^2 = 3$

c) $x + 2y + z = 1$

d) $z - x^2 - y^2 = 0$

e) $e^x + 2y + \sin(z) = 1$

f) $xy + 2z + 5y = 2$
**Def.** A linear eqn. in the variables \( x_1, \ldots, x_n \) is an equation that can be written as

\[
A_1 x_1 + A_2 x_2 + \cdots + A_n x_n = b \quad (1)
\]

Where \( A_1, \ldots, A_n \) are real or complex numbers called the "coefficients" and \( b \) is also a real or complex number called the "independent term."

**Solution**

Consider the pairs: i) \((1,1)\), ii) \((2,1)\)

Subst. \((a)\) leads to

\[(1,1): \quad 2(1) - 1 = 1 \quad \checkmark \quad \text{eqn. is satisfied.}\]

\[(2,1): \quad 2(2) - 1 = 3 \neq 1 \quad \text{"is not".}\]

We will say that \((1,1)\) is a "soln" of eqn. \((a)\) and \((2,1)\) is not a "soln."

Sssk! How many solns. does eqn. \((a)\) have?

*All points on the line \( y = 2x - 1 \)* are solutions. In fact, they are all the solutions. We say that all points in \( L \) constitute the "general soln." of eqn. \((a)\).
Def.: A soln. of a lin. eqn. (1)
\[ a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b \]
is a list of \( n \) numbers \( s_1, s_2, \ldots, s_n \) such that eqn. (1) is satisfied when we substitute \( x_1 = s_1, \ldots, x_n = s_n \).
**Linear Systems.**

**Example 1.**
\[
\begin{align*}
2x - y &= 1 \\
x - y &= -1
\end{align*}
\]

(2,3) is a soln. Since
\[
2(2) - 3 = 1 \\ 2 - 3 = -1
\]

but (4,5) is not. Since
\[
2(4) - 5 = 3 
eq 1 \\ 4 - 5 = -1
\]

**Example 2.**
\[
\begin{align*}
2x - y &= 1 \\
2x - y &= -1
\end{align*}
\]

**Example 3.**
\[
\begin{align*}
2x - y &= 1 \\
4x - 2y &= 2
\end{align*}
\]

(2,3) and (3,5) are solns.
but (1,2) is not a soln.

**Geometrical soln.**
Def.: A System of linear equs. is a collection of one or more linear equs. involving the same variable $x_1, x_2, \ldots, x_n$

$$\begin{cases} 
    a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m 
\end{cases}$$

$m$ equations and $n$ variables or unknowns.

Def.: A list of numbers $s_1, s_2, \ldots, s_n$ is a soln. of (1) if this list is a solution of every equation of system (1).

Example 1 - 3 illustrate the following general result:

"Every System of linear equs. has either no solns., exactly one soln. or infinitely many solns."
Def.: A linear syst with one soln. or infinitely many solns. is said to be consistent. Otherwise, it's called inconsistent.

Matrix Representation

Example 4.-

\begin{align*}
2x_1 & -4x_3 = -10 \\
-x_2 + 3x_3 &= 2 \\
3x_1 + 5x_2 + 8x_3 &= -6
\end{align*}

Associated matrix to example 4 is given by

\[
A = \begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 3 \\ 3 & 5 & 8 \end{pmatrix}
\]

It's called Coefficient matrix. Another important matrix is the Augmented matrix:

\[
A = \begin{pmatrix} 2 & 0 & -4 & -10 \\ 0 & 1 & 3 & 2 \\ 3 & 5 & 8 & -6 \end{pmatrix}
\]
Solving a linear system

Replace original syst by another syst. easier to solve.

Three basic operations are performed on original syst.

1. Scaling. Multiply by a nonzero constant
2. Replacement. add a multiple of one equation to another
3. Interchange. Interchange two equations.

By an equivalent system we mean a new system that has the same set of solns as the original one.

Work on transparency, Example 4.
Example 4.

\[ 2x_1 - 4x_3 = -10 \]
\[ x_2 + 3x_3 = 2 \]
\[ 3x_1 + 5x_2 + 8x_3 = -6 \]

1. Multiply 1st equ. by \( \frac{1}{2} \).

\[ x_1 - 2x_3 = -5 \]
\[ x_2 + 3x_3 = 2 \]
\[ 3x_1 + 5x_2 + 8x_3 = -6 \]

2. Add \( -3 \times (\text{equ. 1}) \) to equ. 3 to eliminate \( x_1 \) coeff.

\[ x_1 - 2x_3 = -5 \]
\[ x_2 + 3x_3 = 2 \]
\[ 5x_2 + 14x_3 = 9 \]

3. \(-5 \times (\text{equ. 2}) + \text{equ. 3}\)

\[ x_1 - 2x_3 = -5 \]
\[ x_2 + 3x_3 = 2 \]
\[ -x_3 = -1 \]
(4) \((-1) \times (\text{eq. 3})\)

\[
\begin{align*}
  x_1 - 2x_3 &= -5 \\
  x_2 + 3x_3 &= 2 \\
  x_3 &= 1
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 0 & -2 & -5 \\
  0 & 1 & 3 & 2 \\
  0 & 0 & 1 & 1
\end{pmatrix}
\]

(5) \((-3) \times (\text{eq. 3}) + \text{eq. 2}\)

\[
\begin{align*}
  x_1 - 2x_3 &= -5 \\
  x_2 &= -1 \\
  x_3 &= 1
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 0 & -2 & -5 \\
  0 & 1 & 0 & -1 \\
  0 & 0 & 1 & 1
\end{pmatrix}
\]

(6) \(2 \times (\text{eq. 3}) + \text{eq. 1}\)

\[
\begin{align*}
  x_1 &= -3 \\
  x_2 &= -1 \\
  x_3 &= 1
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 0 & 0 & -3 \\
  0 & 1 & 0 & -1 \\
  0 & 0 & 1 & 1
\end{pmatrix}
\]

Sohm: \((x_1, x_2, x_3) = (-3, -1, 1)\).
Example 4. (Summary)

\[\begin{align*}
2x_1 - 4x_3 &= -10 \\
x_2 + 3x_3 &= 2 \\
3x_1 + 5x_2 + 8x_3 &= -6
\end{align*}\]

\[A = \begin{pmatrix}
2 & 0 & -4 & -10 \\
0 & 1 & 3 & 2 \\
3 & 5 & 8 & -6
\end{pmatrix}\]

Basic operations or row operations

\[\begin{align*}
x_1 - 2x_3 &= -5 \\
x_2 + 3x_3 &= 2 \\
x_3 &= 1
\end{align*}\]

\[B = \begin{pmatrix}
1 & 0 & -2 & -5 \\
0 & 1 & 3 & 2 \\
0 & 0 & 1 & 1
\end{pmatrix}\]

Further row operations

\[\begin{align*}
x_1 &= -3 \\
x_2 &= -1 \\
x_3 &= 1
\end{align*}\]

\[C = \begin{pmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1
\end{pmatrix}\]
The three basic operations on the equations of the system correspond to three elementary row operations on the augmented matrix:

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

Def: We will say that two matrices are row equivalent if there is a sequence of elementary row operations that transform one matrix into the other.

Remark: Row operations are reversible.

If the augmented matrices of two linear systems are row equivalent then the two systems have the same solution set.
Exercise #12

(1) \( x_1 - 5x_2 + 4x_3 = -3 \)
(2) \( 2x_1 - 7x_2 + 3x_3 = -2 \)
(3) \( -2x_1 + x_2 + 7x_3 = -1 \)

\[
\begin{pmatrix}
1 & -5 & 4 & -3 \\
2 & -7 & 3 & -2 \\
-2 & 1 & 7 & -1
\end{pmatrix}
\]

① \( Eq. (1) \times (-2) + Eq. (2) \) and \( Eq. (1) \times (2) + Eq. (3) \)

\[
\begin{align*}
x_1 - 5x_1 + 4x_3 &= -3 \\
3x_1 - 5x_3 &= 4 \\
-9x_1 + 15x_3 &= -7
\end{align*}
\]

② \( Eq. (2) \times (3) + Eq. (3) \)

\[
\begin{align*}
x_1 - 5x_1 + 4x_3 &= -3 \\
3x_1 - 5x_3 &= 4
\end{align*}
\]

\[
\begin{pmatrix}
1 & -5 & 4 & -3 \\
0 & 3 & -5 & 4 \\
0 & 0 & 0 & 5
\end{pmatrix}
\]

Therefore, the system (1)-(3) is "inconsistent." It means, it has no solution.
Two Fundamental Questions About Linear Systems

1. Is the System consistent?

2. Is the Solution unique?

Consistent means there is at least one Solution.

Unique means if a solution exists that Solution is the only one.