9.5 Singular Value Decomposition

Extension of the Diagonalization theory for square matrices \(mn \times mn\) to non-square matrices \(mn \times n\).

In previous sections, we have found that

I) If \(A\) is Symm.

\[ A = PDP^T \quad \text{(Eigenvalue Decomp.)} \]

\[ D: \text{diagonal and columns of } P \text{ are eigenvectors of } A. \]

\[ P \text{ is an orthogonal matrix.} \]

II) If \(A\) is nonsym.

\[ A = PHPT \quad \text{Hessenberg Decomp.} \]

\[ P \text{ is an orthogonal matrix and } H \text{ is upper Hessenberg.} \]

III) If eigenvalues of \(A\) are real

\[ A = PSP^T \]

\[ P \text{ is an orthogonal and } S \text{ is upper triangular.} \]

Importance of orthogonal matrix in numerical computation

If \(\|x - \hat{x}\| \leq e\)

\[ \Rightarrow \|P(x - \hat{x})\| = \|x - \hat{x}\| \leq e \quad \text{errors are not magnified.} \]
In this section, we will find that for an arbitrary $A_{m \times n}$ matrix

$$A = U \Sigma V^T$$

$U$, $V$ orthog. and $\Sigma$ is line-deg. for nonsquare matrices.

**Thm 9.5.1** 

A $m \times n$ matrix

a) $A$ and $A^T A$ have same null space.

b) $A$ and $A^T A$ have row space.

c) $A^T$ and $A^T A$ have column space.

d) $A$ and $A^T A$ have rank.

**Proof:**

$\Rightarrow$ We want to prove

if $A \tilde{x} = \tilde{0} \Rightarrow (A^T A) \tilde{x} = \tilde{0}$

In fact, if $A \tilde{x} = \tilde{0} \Rightarrow (A^T A) \tilde{x} = A^T (A \tilde{x}) = A^T (\tilde{0}) = \tilde{0}$

$\Leftarrow$ We want to prove:

if $A^T A \tilde{x} = \tilde{0} \Rightarrow A \tilde{x} = \tilde{0}$.

In fact, $A^T A \tilde{x} = \tilde{0} \Rightarrow \tilde{x} \in \text{Null}(A^T A) = [\text{Null}(A^T A)]^\perp = [\text{Col}(A^T A)]^\perp$

Also, $A^T A \tilde{x} \in \text{Col}(A^T A) \Rightarrow \tilde{x}, A^T A \tilde{x} = \tilde{0}$.

Or $0 = (A^T A) \tilde{x} = \tilde{x}^T (A^T A) \tilde{x} = \tilde{x}^T A^T A \tilde{x} = (x^T x)(A \tilde{x}) = (A \tilde{x})^T A \tilde{x} = A \tilde{x} \cdot A \tilde{x} = ||A \tilde{x}||^2$

Thus, $||A \tilde{x}||^2 = 0 \Rightarrow A \tilde{x} = \tilde{0}$. \(\checkmark\)
Thm 9.5.2 A m×n matrix, then

a) $A^T A$ is orthogonally diagonalizable

b) The eigenvalues of $A^T A$ are nonnegative

Proof:

a) $A^T A$ is symmetric $(A^T A)^T = A^T (A^T)^T = A^T A$ then by theorem 7.2.1 $A^T A$ is orthogonal.

b) First notice that

$$A^T A \mathbf{v} = \mathbf{v}^T A^T A \mathbf{v} = (A \mathbf{v})^T A \mathbf{v} = A \mathbf{v} \cdot A \mathbf{v} = \|A \mathbf{v}\|^2$$

Then

$$\|A \mathbf{v}\|^2 = (A^T A) \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot A^T \mathbf{v} = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = \lambda.$$

If $\mathbf{v}$ is a unit eigenvector of $A^T A$ with eigenvalue $\lambda$,

$$\Rightarrow \lambda \geq 0.$$

Def. For a $m \times n$ matrix

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A^T A$, then

the real numbers

$$\sigma_1 = \sqrt{\lambda_1}, \quad \sigma_2 = \sqrt{\lambda_2}, \quad \ldots, \quad \sigma_n = \sqrt{\lambda_n}$$

are called singular values of $A$. 
Before turning to the main result in this section, we will find it useful to extend the notion of a "main diagonal" to matrices that are not square. We define the main diagonal of an \( m \times n \) matrix to be the line of entries shown in Figure 9.5.1—it starts at the upper left corner and extends diagonally as far as it can go. We will refer to the entries on the main diagonal as the diagonal entries.

We are now ready to consider the main result in this section, which is concerned with a specific way of factoring a general \( m \times n \) matrix \( A \). This factorization, called singular value decomposition (abbreviated SVD) will be given in two forms, a brief form that captures the main idea, and an expanded form that spells out the details. The proof is given at the end of this section.

**THEOREM 9.5.3 Singular Value Decomposition**

If \( A \) is an \( m \times n \) matrix, then \( A \) can be expressed in the form

\[
A = U \Sigma V^T
\]

where \( U \) and \( V \) are orthogonal matrices and \( \Sigma \) is an \( m \times n \) matrix whose diagonal entries are the singular values of \( A \) and whose other entries are zero.

**THEOREM 9.5.4 Singular Value Decomposition (Expanded Form)**

If \( A \) is an \( m \times n \) matrix of rank \( k \), then \( A \) can be factored as

\[
A = U \Sigma V^T = [u_1 \ u_2 \ \cdots \ u_k \ u_{k+1} \ \cdots \ u_n]
\]

\[
\begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_k \\
0_{(m-k)\times k} & 0_{(m-k)\times (n-k)}
\end{bmatrix}
\begin{bmatrix}
v_1^T \\
v_2^T \\
\vdots \\
v_k^T \\
v_{k+1}^T \\
\vdots \\
v_n^T
\end{bmatrix}
\]

in which \( U, \Sigma, \) and \( V \) have sizes \( m \times m \), \( m \times n \), and \( n \times n \), respectively, and in which

(a) \( V = [v_1 \ v_2 \ \cdots \ v_n] \) orthogonally diagonalizes \( A^T A \).

(b) The nonzero diagonal entries of \( \Sigma \) are \( \sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \ldots, \sigma_k = \sqrt{\lambda_k} \), where \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are the nonzero eigenvalues of \( A^T A \) corresponding to the column vectors of \( V \).

(c) The column vectors of \( V \) are ordered so that \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0 \).

(d) \( u_i = \frac{A v_i}{\|Av_i\|} = \frac{1}{\sigma_i} A v_i \quad (i = 1, 2, \ldots, k) \)

(e) \( \{u_1, u_2, \ldots, u_k\} \) is an orthonormal basis for \( \text{col}(A) \).

(f) \( \{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_m\} \) is an extension of \( \{u_1, u_2, \ldots, u_k\} \) to an orthonormal basis for \( R^n \).
**Def.** For an $m \times n$ matrix $A$, we define the main diagonal to be the line of entries that starts at the upper left corner and extends diagonally as far as it can go. The entries on this main diagonal are called diagonal entries. (See Fig 9.5.1)

For the theorem look at page 509 book.

**Proof.**

Hypothesis are

1. A $m \times n$ matrix
2. $\text{rank}(A) = k$

The first thing to consider is that: 

\[
\begin{align*}
A^T A &= V D V^T \\
\text{where } V &\text{ is orthogonal, and the column vectors of } V \\
&\text{are orthonormal eigenvectors of } A^T A,
\end{align*}
\]

\[
V = [v_1 \ v_2 \ \ldots \ v_n]
\]

Corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that may be repeated and also some of them may be zero.

Since \( \text{rank}(A) = k \), then (thm 9.5.1) \( \text{rank}(A^T A) = k \).

This also implies that \( \text{rank}(D) = k \), because $D$ is similar to $A^T A$. 
Then,

\[ D = \begin{bmatrix}
\lambda_1 & & & & \\
& \lambda_2 & & & \\
& & \ddots & & \\
& & & \lambda_k & \\
& & & & 0
\end{bmatrix} \quad \text{where} \quad \lambda_j \neq 0, \ j = 1, \ldots, k.

From the orthogonality of \( \tilde{v}_j \)'s,

it can be shown that the set \( \{ A \tilde{v}_1, \ldots, A \tilde{v}_n \} \) is a set of orthogonal vectors. In fact,

\[ A \tilde{v}_i \cdot A \tilde{v}_j = \tilde{v}_j^T A^T A \tilde{v}_i = (\tilde{v}_j^T A^T A) \tilde{v}_i = 0.
\]

\[ \tilde{v}_i \cdot (A^T A)^T \tilde{v}_j = \tilde{v}_i \cdot A^T A \tilde{v}_j = \tilde{v}_i \cdot \lambda_j \tilde{v}_j = \lambda_j \tilde{v}_i \cdot \tilde{v}_j = 0 \quad \text{because} \quad \tilde{v}_i \cdot \tilde{v}_j = 0
\]

We showed that

\[ \| A \tilde{v}_i \|^2 = \lambda_i, \quad i = 1, 2, \ldots, K, \ldots, n
\]

\[ \Rightarrow \| A \tilde{v}_i \| \neq 0, \quad i = 1, 2, \ldots, K
\]

Then \( S = \{ A \tilde{v}_1, \ldots, A \tilde{v}_n \} \) is an orthogonal set of nonzero vectors and because each \( A \tilde{v}_i \in \text{Col}(A) \Rightarrow S \subseteq \text{Col}(A)
\]

Also, \( \text{rank}(A) = \text{dim}(\text{col}(A)) = K \), then \( S \) is a basis for \( \text{Col}(A) \). An orthonormal basis for \( \text{Col}(A) \) is given by

\[ \{ \tilde{u}_1, \ldots, \tilde{u}_k \} = \left\{ \frac{A \tilde{v}_1}{\| A \tilde{v}_1 \|}, \frac{A \tilde{v}_2}{\| A \tilde{v}_2 \|}, \ldots, \frac{A \tilde{v}_k}{\| A \tilde{v}_k \|} \right\}
\]

\[ \frac{1}{\sqrt{\lambda_1}} A \tilde{v}_1, \frac{1}{\sqrt{\lambda_2}} A \tilde{v}_2, \ldots, \frac{1}{\sqrt{\lambda_k}} A \tilde{v}_k
\]
or \[
\begin{align*}
\hat{A}_{ik} &= \sqrt{\lambda_i} \hat{u}_i = \sigma_i \hat{u}_i, \quad \ldots \quad A \hat{v}_n &= \sigma_n \hat{v}_n. \\
\end{align*}
\] (\star)

Since \( A_{m \times n} (A_{vi})_{m \times 1} \in \mathbb{R}^m, \ i = 1, 2, \ldots, n \)

Thus, the set \( \{ \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n \} \subset \text{Col}(A) \)
and can be extended to an orthonormal basis \( \{ \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_k, \hat{u}_{k+1}, \ldots, \hat{u}_m \} \)

define \( U \) as the orthogonal matrix
\[
U = \begin{bmatrix}
\hat{u}_1 & \hat{u}_2 & \ldots & \hat{u}_k & \hat{u}_{k+1} & \ldots & \hat{u}_m
\end{bmatrix}_{m \times m}
\]

and let \( \Sigma \) be the \( m \times n \) matrix
\[
\begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_r
\end{bmatrix}_{r \times r}
\]

Then
\[
U \Sigma = \begin{bmatrix}
\sigma_1 \hat{u}_1 & \sigma_2 \hat{u}_2 & \cdots & \sigma_r \hat{u}_r & 0 & \cdots & 0
\end{bmatrix}_{m \times n} = \\
\begin{bmatrix}
A \hat{v}_1 & A \hat{v}_2 & \cdots & A \hat{v}_k & A \hat{v}_{k+1} & \cdots & A \hat{v}_n
\end{bmatrix}_{m \times n} = AV
\]
Problem #9) Find S.V.D. of
\[ A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \]

1) Find \( A^T A \)
\[ A^T A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \]

Diagonalizes \( A^T A \)

Orthogonally

\[ |\lambda I - A^T A| = \begin{vmatrix} \lambda - 9 & 9 \\ 9 & \lambda - 9 \end{vmatrix} = 0 \iff (\lambda - 9)^2 - 81 = \lambda^2 - 18 \lambda = 0 \]
\[ \lambda (\lambda - 18) = 0 \]
\[ \lambda_1 = 18, \quad \lambda_2 = 0. \]

Eigenvectors for \( \lambda_1 = 18 \)

\[ \begin{bmatrix} 9 & 9 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[ \begin{bmatrix} x_1 = -x_2 \\ x_1 = -x_2 \end{bmatrix} \]
\[ \chi = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

So \[ \chi_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \] eigenvector associated to \( \lambda_1 = 18 \).
For $\lambda_2 = 0$,

\[
\begin{bmatrix}
-9 & 9 \\
9 & -9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=egin{bmatrix}
0 \\
x_2
\end{bmatrix}
\]

Then $\mathbf{v}_2 = \begin{bmatrix}
0 \\
x_2
\end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 0$.

Therefore, if we define

\[
P = \begin{bmatrix}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

then $\mathbf{A}^T \mathbf{A}$ can be orthogonally diagonalized as

\[
P \mathbf{A}^T \mathbf{A} \mathbf{P}^T = \mathbf{D}
\]

\[
P = \begin{bmatrix}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix}
-9 & 9 \\
9 & -9
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
= \begin{bmatrix}
18 & 0 \\
0 & 0
\end{bmatrix}
\]

3) Construct $\mathbf{V}$ and $\mathbf{V}^T$ of the Singular Value Decomposition theorem.

According to Thm 9.5.4,

\[
\mathbf{V} = \mathbf{P} = \begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

and

\[
\Sigma = \begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix}
= \begin{bmatrix}
\sqrt{18} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
3\sqrt{2} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
To construct the matrix \( \mathbf{U} \), we need to consider \( \mathbf{A} \mathbf{v}_i \) for \( \mathbf{v}_i \) such that \( \mathbf{A} \mathbf{v}_i \neq \mathbf{0} \).

Thus,
\[
\mathbf{A} \mathbf{v}_1 = \begin{bmatrix} -2 & 2 \\ -1 & 2 \\ \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 4 \sqrt{2} \\ 2 \sqrt{2} \\ -4 \sqrt{2} \end{bmatrix}
\]

Also,
\[
||\mathbf{A} \mathbf{v}_1|| = \sqrt{\frac{16}{2} + \frac{4}{2} + \frac{1}{2}} = \sqrt{\frac{6}{2}} = \frac{6}{\sqrt{2}} = \frac{6 \sqrt{2}}{2} = 3 \sqrt{2}
\]

\[
\Rightarrow \frac{\mathbf{A} \mathbf{v}_1}{||\mathbf{A} \mathbf{v}_1||} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}
\]

According to 9.5.4, \( \mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \) then \( \mathbf{u}_1 = \left( \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \).

Since \( \mathbf{A} \mathbf{v}_2 = 0 \mathbf{v}_2 = \mathbf{0} \), therefore, there is not a \( \mathbf{u}_2 \) obtained this way.

4) To complete the construction of an orthogonal matrix \( \mathbf{U} \)

we need to extend the set \( \mathbf{S} = \{ \mathbf{u}_1 \} \) to an orthonormal basis of \( \mathbb{R}^3 \). (Notice that this is not unique!)

This is not hard, just consider
\[
\mathbf{u}_1 \cdot \mathbf{u}_2 = 0 \iff \left( \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \cdot (u_{21}, u_{22}, u_{23}) = 0
\]

or
\[
\frac{2}{3} \mathbf{u}_{21} + \frac{1}{3} \mathbf{u}_{22} - \frac{2}{3} \mathbf{u}_{23} = 0 \Rightarrow \mathbf{u}_{21} = \mathbf{u}_{22} - \frac{1}{2} \mathbf{u}_{23}
\]

or
\[
\mathbf{u}_2 = \begin{bmatrix} u_{21} - \frac{1}{2} u_{22} \\ u_{22} \\ u_{23} \end{bmatrix} = \begin{bmatrix} u_{23} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} u_{22} \\ u_{22} \\ 0 \end{bmatrix} = \begin{bmatrix} u_{23} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]
One possibility is to choose $U_{23} = 1$, $U_{22} = 0$.

Then, $\vec{U}_2' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, obviously not a unit vector.

Thus, $\vec{U}_2 = \begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix}$, $\hat{u}_1$, $\hat{u}_2$.

That way we have an orthonormal set $S' = \left\{ \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix} \right\}$.

Our purpose is to extend this to a basis in $\mathbb{R}^3$ which is orthonormal. So, we look for a 3rd vector $\vec{U}_3'$ such that

$\vec{U}_1 \perp \vec{U}_3' \iff \begin{pmatrix} \frac{2}{3} U_{31} + \frac{1}{3} U_{32} - \frac{2}{3} U_{33} = 0 \\ \frac{1}{\sqrt{2}} U_{31} + 0 U_{32} + \frac{1}{\sqrt{2}} U_{33} = 0 \end{pmatrix}$

\[ \Rightarrow \begin{cases} U_{31} = -U_{33} \\ 4U_{32} + U_{33} = 0 \Rightarrow U_{32} = -\frac{1}{4} U_{33} \end{cases} \]

\[ \Rightarrow \vec{U}_3' = \begin{pmatrix} -U_{33} \\ -4U_{33} \\ U_{33} \end{pmatrix} = U_{33} \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \Rightarrow \vec{U}_3 = \begin{pmatrix} -\frac{1}{\sqrt{8}} \\ -\frac{4}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \end{pmatrix} \]

or $\vec{U}_3 = \begin{pmatrix} -\frac{\sqrt{2}}{6} \\ -\frac{2\sqrt{2}}{3} \\ \frac{\sqrt{6}}{6} \end{pmatrix}$.
Finally, the orthogonal matrix $\mathbf{U}$ sought is formed by orthonormal vectors

$$
\mathbf{U} = [\hat{u}_1 \hat{u}_2 \hat{u}_3],
$$

where

$$
\hat{u}_i = \frac{A \hat{u}_i}{\|A \hat{u}_i\|}
$$

and

$$
A = \mathbf{U} \Sigma \mathbf{V}^T
$$

where

$$
\mathbf{U} = \begin{bmatrix}
\frac{2}{3} & \frac{1}{2} & -\frac{\sqrt{6}}{2} \\
\frac{1}{3} & 0 & -\frac{\sqrt{2}}{3} \\
-\frac{2}{3} & \frac{1}{2} & \frac{\sqrt{6}}{3}
\end{bmatrix}, \quad
\Sigma = \begin{bmatrix}
\sqrt{6} & 0 \\
0 & 0
\end{bmatrix}
$$

$$
\mathbf{V} = \begin{bmatrix}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
$$

Notice, that the matrix $\mathbf{U}$ is not unique. There are infinitely many ways to complete $S = \{\mathbf{u}_i\}$ as an orthonormal basis for $\mathbb{R}^3$. 
The Singular Value Decomposition reduces to the orthogonal diagonalization of a matrix $A$ if $A$ is symmetric.

In fact, for any matrix $A$, the Singular Value Decomposition is given by

$$A = U \Sigma V^T$$

where $\Sigma$ has nonzero entries only along its main diagonal, which are the Singular values of $A$, or

$$\sigma_i = \sqrt{\lambda_i}, \quad \text{where} \quad A^T A \tilde{v}_i = \lambda_i \tilde{v}_i, \quad i = 1, 2, \ldots, k \leq n$$

Also, $V = \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \ldots & \tilde{v}_n \end{bmatrix}$ are the eigenvectors corresponding to $\lambda_i$, $i = 1, 2, \ldots, n$.

and $U = \begin{bmatrix} \tilde{u}_1 & \tilde{u}_2 & \ldots & \tilde{u}_m \end{bmatrix} = \begin{bmatrix} \frac{A \tilde{v}_1}{\|A \tilde{v}_1\|} & \frac{A \tilde{v}_2}{\|A \tilde{v}_2\|} & \ldots & \frac{A \tilde{v}_k}{\|A \tilde{v}_k\|} & \frac{A \tilde{v}_{k+1}}{\|A \tilde{v}_{k+1}\|} & \ldots & \frac{A \tilde{v}_m}{\|A \tilde{v}_m\|} \end{bmatrix}$

Now, if $A$ is Symm. $A = A^T$

then, $A^T A \tilde{v}_i = \lambda_i \tilde{v}_i \Rightarrow A^2 \tilde{v}_i = \lambda_i \tilde{v}_i \Rightarrow \sqrt{\lambda_i} \tilde{v}_i$ is an eigenvalue for $A$ with the same corresponding eigenvector $\tilde{v}_i$.

$$\Rightarrow \begin{bmatrix} A \tilde{v}_i &=& \sqrt{\lambda_i} \tilde{v}_i \end{bmatrix} \Rightarrow \|A \tilde{v}_i\| = \sqrt{\lambda_i} \|\tilde{v}_i\| = \sqrt{\lambda_i} = \sigma_i, \quad i = 1, 2, \ldots, k$$

$$\Rightarrow U = \begin{bmatrix} \frac{\sqrt{\lambda_1} \tilde{v}_1}{\sqrt{\lambda_1}} & \frac{\sqrt{\lambda_2} \tilde{v}_2}{\sqrt{\lambda_2}} & \ldots & \frac{\sqrt{\lambda_k} \tilde{v}_k}{\sqrt{\lambda_k}} & \tilde{u}_{k+1} & \ldots & \tilde{u}_n \end{bmatrix}$$

or $U = \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \ldots & \tilde{v}_k & \tilde{u}_{k+1} & \ldots & \tilde{u}_n \end{bmatrix}$
A possible orthonormal $U$ is given by

$$U = \begin{bmatrix} \hat{v}_1 & \cdots & \hat{v}_k & \hat{v}_{k+1} & \cdots & \hat{v}_n \end{bmatrix}$$

Large eigenvectors correspond to $\lambda = 0$ eigenvalue.

Therefore, the matrix $A$ can be decomposed as

$$A = U \Sigma V^T$$

Where

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix} = \begin{bmatrix} \hat{v}_{\lambda_1} & \hat{v}_{\lambda_2} & \cdots & \hat{v}_{\lambda_k} \end{bmatrix}$$

$$= \mathbf{D}$$

Which is the orthogonal diagonalization of $A$. 