DEFINITION 1 Let $V$ be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By **addition** we mean a rule for associating with each pair of objects $u$ and $v$ in $V$ an object $u + v$, called the **sum** of $u$ and $v$; by **scalar multiplication** we mean a rule for associating with each scalar $k$ and each object $u$ in $V$ an object $ku$, called the **scalar multiple** of $u$ by $k$. If the following axioms are satisfied by all objects $u, v, w$ in $V$ and all scalars $k$ and $m$, then we call $V$ a **vector space** and we call the objects in $V$ **vectors**.

1. If $u$ and $v$ are objects in $V$, then $u + v$ is in $V$.
2. $u + v = v + u$
3. $u + (v + w) = (u + v) + w$
4. There is an object $0$ in $V$, called a **zero vector** for $V$, such that $0 + u = u + 0 = u$ for all $u$ in $V$.
5. For each $u$ in $V$, there is an object $-u$ in $V$, called a **negative** of $u$, such that $u + (-u) = (-u) + u = 0$.
6. If $k$ is any scalar and $u$ is any object in $V$, then $ku$ is in $V$.
7. $k(u + v) = ku + kv$
8. $(k + m)u = ku + mu$
9. $k(mu) = (km)(u)$
10. $1u = u$

Vector space scalars can be real numbers or complex numbers. Vector spaces with real scalars are called **real vector spaces** and those with complex scalars are called **complex vector spaces**. For now we will be concerned exclusively with real vector spaces. We will consider complex vector spaces later.

Observe that the definition of a vector space does not specify the nature of the vectors or the operations. Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on $\mathbb{R}^n$. The only requirement is that the ten vector space axioms be satisfied. In the examples that follow we will use four basic steps to show that a set with two operations is a vector space.

**To Show that a Set with Two Operations is a Vector Space**

**Step 1.** Identify the set $V$ of objects that will become vectors.

**Step 2.** Identify the addition and scalar multiplication operations on $V$.

**Step 3.** Verify Axioms 1 and 6; that is, adding two vectors in $V$ produces a vector in $V$, and multiplying a vector in $V$ by a scalar also produces a vector in $V$. Axiom 1 is called **closure under addition**, and Axiom 6 is called **closure under scalar multiplication**.

**Step 4.** Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

**Historical Note** The notion of an "abstract vector space" evolved over many years and had many contributors. The idea crystallized with the work of the German mathematician H. G. Grassmann, who published a paper in 1862 in which he considered abstract systems of unspecified elements on which he defined formal operations of addition and scalar multiplication.

Our first example will be a vector space. Since $A_i$, $B_i$, $C_i$, ..., will have to be the same object. Since $A_i$, $B_i$, $C_i$, ..., will have to be the same object.

**Example** Let $V$ consist of all sequences $u_1, u_2, u_3, \ldots$ for all scalars $k$.

Our second example is the vector space $\mathbb{R}^n$. It should be noted that the axioms hold for all vectors $u_1, u_2, u_3, \ldots, u_n$.

**Example** Let $V = \mathbb{R}^n$, an $n$-dimensional vector space, where the operations of addition and scalar multiplication are defined as:

- $u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$
- $ku = (ku_1, ku_2, \ldots, ku_n)$

The set $V = \mathbb{R}^n$, where $n \geq 2$, has a finite number of operations producing $n$ vectors, $n = 3, 4, 5, 7, 8, 9, \ldots$.

Our next example is the vector space $\mathbb{R}^n$, where $n$ is the number of components.

**Example** Let $V$ consist of all sequences $u_1, u_2, u_3, \ldots$ in which $u_1, u_2, u_3, \ldots$ is an infinite sequence.

We leave it as an exercise to define addition and scalar multiplication for this space.
Example 1. \( S = \{ \hat{0}, \hat{0} \}, \quad \hat{0} + \hat{0} = \hat{0}, \quad \mu \hat{0} = \hat{0} \)

Example 2. - Matrix: \( M_{2 \times 2} \)
\[
\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
\]

**Def.** \( \mathbf{U} + \mathbf{V} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \in \mathbb{M}_{2 \times 2} \)

\( k \mathbf{U} = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \in \mathbb{M}_{2 \times 2} \)

Easy to Verify - All Axioms: \( \hat{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \)
All entries are real numbers and the real numbers satisfy all the axioms. \( -\mathbf{U} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} \)

Example 3. - Matrix: \( M_{m \times n} \)

Example 4. - \( F:(-\infty, \infty) = \left\{ f : \mathbb{R} \to \mathbb{R} \right\} \)
\( (f+g)(x) = f(x) + g(x) \in \mathbb{R} \quad \text{w.r.t.} \quad F:(-\infty, \infty) \)
\( (k \cdot f)(x) = k \cdot f(x) \in \mathbb{R} \quad \text{w.r.t.} \quad F:(-\infty, \infty) \)

As we can see, \( \| f + g \| = ? \)
Other possibilities: \( F([a,b]) \) or \( F(a,b) \)

Since Values are real numbers and easy to prove all the other properties.
Example 5. - Not a vector space.

\[ V = \mathbb{R}^2 \] but operations defined

a) \[ \tilde{u} + \tilde{v} = (u_1 + v_1, u_2 + v_2) \]
b) \[ k\tilde{u} = (ku_1, 0) \]

Everything is satisfied except which one?

\[ 1\tilde{u} = 1(u_1, u_2) = (1u_1, 0) = (u_1, 0) \neq \tilde{u} \]

Example 6. - Every plane through the origin a vector space.

\[ \mathcal{P} = \{ \tilde{x} = (x_1, y_1, z) : ax + by + cz = 0 \} \]

Under usual operations.

Example 6. - Not a vector space any plane not passing through the origin.

all \( \tilde{x} \) satisfying

\[ ax + by + cz = d \neq 0 \]

(0,0,0) is not \( \mu \in \mathcal{P} \).

also \( \tilde{x} = (x_1, x_2, x_3) \in \mathcal{P} \)

\[ \tilde{y} = (y_1, y_2, y_3) \in \mathcal{P} \]

\[ \tilde{x} + \tilde{y} \text{ not } \mu \in \mathcal{P}. \]

In fact, \[ a(x_1 + y_1) + b(x_2 + y_2) + c(x_3 + y_3) = (ax_1 + ... + cx_3) + (ay_1 + ... + cy_3) = d + d = 2d \neq d \]
Example 8 (Look)  Interesting Vector Space

\[ V = \mathbb{R} \]

Operations:
1) \( u + v = u v \)
2) \( ku = u^k \)

Then \( 1 + 1 = 1.1 = 1 \) and \( 2(1) = 1^2 = 1 \)

All axioms are verified!

i) The vector is 1 or \( \vec{0} = 1 \).

ii) For \( u \), \( -u = \frac{1}{u} \)

because \( u + (-u) = u \frac{1}{u} = 1 = \vec{0} \).

iii) \( k(u + v) = (uv)^k = u^k v^k = (ku + kv) \)

iv) \( u + (v + w) = (u + v) + w \)

Since \( u + (v + w) = u(v + w) = u(vw) = (uv)w = (u + v)w = (u + v) + w \)
Thm 5.1.1 \( V \) is a vector space, \( \vec{u} \) and \( \vec{v} \) are in \( V \) and \( k \) is scalar, then

a) \( 0\vec{u} = \vec{0} \), b) \( k\vec{0} = \vec{0} \), c) \( (-1)\vec{u} = -\vec{u} \).

d) If \( k\vec{u} = \vec{0} \), then \( k = 0 \) or \( \vec{u} = \vec{0} \).

**Proof**

b) \( k\vec{0} = \vec{0} \)

\[
\begin{align*}
\vec{k}\vec{0} &= \vec{k}(\vec{0} + \vec{0}) \\
&= \vec{k}\vec{0} + \vec{k}\vec{0} \quad (\star)
\end{align*}
\]

Now, \(-\vec{k}\vec{0}\) exists (axiom 5). Adding it to both sides of equation (\(\star\)).

\[
\begin{align*}
\vec{k}\vec{0} + (-\vec{k}\vec{0}) &= (\vec{k}\vec{0} + \vec{k}\vec{0}) + (-\vec{k}\vec{0}) \\
\text{A} &= \text{B}
\end{align*}
\]

Now,

\[
\begin{align*}
\text{A} &= \vec{k}\vec{0} + (-\vec{k}\vec{0}) \quad (1) \\
\text{and} \quad \text{B} &= \vec{k}\vec{0} + (\vec{k}\vec{0} + (-\vec{k}\vec{0})) \quad (2) \\
&= \vec{k}\vec{0} + \vec{0} = k\vec{0} \quad (3)
\end{align*}
\]

Therefore, \( \text{A} = \text{B} \) \( \iff \) \( k\vec{0} = \vec{0} \) \( \checkmark \)
(Good to emphasize logic)

a) If \( k\hat{u} = \hat{o} \), then \( k = 0 \) or \( \hat{u} = \hat{o} \).

Proof:

Assume \( k\hat{u} = \hat{o} \), \( (1) \)

then there are two possibilities for \( k \):

a) \( k = 0 \) and the theorem is proved.

or

b) \( k \neq 0 \), then \( \frac{1}{k} \) is a scalar too

and multiplying both sides of (1) by it

\[
\frac{1}{k} (k\hat{u}) = \frac{1}{k} \hat{o}
\]

\[
\frac{A}{B}
\]

Now, \( A \overset{?}{=} \left( \frac{1}{k} \right) \hat{u} = \hat{u} \overset{\text{?}}{=} \hat{u} \)

and \( B = \frac{1}{k} \hat{o} \overset{\text{?}}{=} \hat{o} \)

Thus, \( A \overset{?}{=} B \Rightarrow \hat{u} = \hat{o} \).
Uniqueness Theorem.

Thus: Given a vector \( \vec{u} \) in a vector space \( V \),
the negative vector \(-\vec{u}\) is unique.

Proof: If there were another vector \( \vec{r} \) in \( V \)
such that
\[
\vec{u} + \vec{r} = \vec{0}, \quad (1)
\]
then adding \(-\vec{u}\) to both sides of (1)
\[
\frac{-\vec{u} + (\vec{u} + \vec{r})}{\text{A}} = \frac{-\vec{u} + \vec{0}}{\text{B}}
\]

Now,
\[
\text{A} = -\vec{u} + (\vec{u} + \vec{r}) = (\vec{u} + \vec{u}) + \vec{r} = \vec{0} + \vec{r} = \vec{r} \quad (3)
\]
and
\[
\text{B} = -\vec{u} + \vec{0} = -\vec{u} \quad (4)
\]

Therefore \( \text{A} = \text{B} \) \( \Rightarrow \) \( \vec{r} = -\vec{u} \)

And as a consequence \(-\vec{u}\) is unique.
\[ S = \{ \text{moon, sun} \} \]

Two elements set

It can't be a vector space.

**Proof:** One of the two elements should be the \( \vec{0} \) vector.

If \( \vec{0} = \text{Sun} \)

then we can define addition as

a) \( \text{moon} + \text{sun} = \text{moon} \)
b) \( \text{sun} + \text{sun} = \text{sun} \)
c) \( \text{moon} + \text{moon} = \text{sun} \) \( \text{(moon needs to have a negative)} \)

Thus the only possibility for \( S = \text{sun} \).

**Scalar product definition**

\( d) \ K \text{sun} = \text{sun}, \text{ for any } K \)
\( e) \ K \text{moon} = \text{moon}, \text{ if } K \neq 0, \text{ and } 0 \text{moon} = \text{sun} \)

Properties for the addition \( d), e), f), g), \text{ and } h) \text{ are satisfied.} \)

Property \( f) \ K \left( \text{sun} + \text{moon} \right) = K \text{moon} = \text{moon}, \text{ if } K \neq 0 \)

And \( K \text{sun} + K \text{moon} = \text{sun} + \text{moon} = \text{moon} \)

If \( K = 0 \), then \( K \left( \text{sun} + \text{moon} \right) = K \text{moon} = 0 \text{moon} = \text{sun} \)

Also \( K \text{sun} + K \text{moon} = 0 \text{ sun} + 0 \text{moon} = \text{sun} + \text{sun} = \text{sun} \).
Property 8: 

\[(k + l) \text{moon} = \text{moon} \]

and 

\[k \text{moon} + l \text{moon} = \text{moon} + \text{moon} = \text{Sun} \]

Therefore, property 8 is not satisfied.

Defining a vector space of three elements.
Theorem 5.21 Assume

1) \( W \subset V \) with one or more vectors

2) \( V \) is a vector space

\( W \) is a subspace of \( V \) if and only if

a) \( W \) is closed under addition. For any \( \tilde{u}, \tilde{v} \in W \), \( \tilde{u} + \tilde{v} \) is also in \( W \).

b) \( W \) is closed under scalar multiplication.

For any \( \tilde{u} \in V \) and a real \( \ell \), \( \ell \tilde{u} \) is in \( V \).

Proof - (\( \rightarrow \)) Trivial \( \odot \) and \( \ominus \) particular case of all field axioms.

(\( \leftarrow \)) All projects (axioms) are inherited, except

4) and 5)

There is a zero, since if \( \tilde{u} \) is in \( W \), then \( 5.11 \Rightarrow 0 \tilde{u} \) is in \( W \), but \( 0 \tilde{u} = \tilde{0} \).

5) For any \( \tilde{u} \in W \), there is \( -\tilde{u} \), since

\( 5.11 \Rightarrow ( -1 ) \tilde{u} \in W \), but \( ( -1 ) \tilde{u} = -\tilde{u} \).
Definition:

\( \mathbf{W} \) is a linear combination of the vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) if

\[ \mathbf{W} = k_1 \mathbf{v}_1 + \cdots + k_r \mathbf{v}_r \]

where \( k_1, \ldots, k_r \) are scalars.

Theorem 5.2.3: If \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) are in \( V \):

a) \( W \) is the set of all possible linear combinations of \( \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \) is a subspace of \( V \).  

b) \( W \) is the smallest subspace of \( V \) containing \( \mathbf{v}_1, \ldots, \mathbf{v}_r \)  

(If \( U \subseteq V \) and \( \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq U \rightarrow W \subseteq U \).

Proof: Easy.

Definition:

\( S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \) and \( W \) as in (a) then 5.2.3

\( W \) is called the space spanned by \( S \).

Notation: \( W = \text{Span}(S) \) or \( W = \text{span} (\mathbf{v}_1, \ldots, \mathbf{v}_r) \).
Theorem 5.2.4

\[ S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r \}, \quad S' = \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \]

two sets of vectors in \( V \) (vector space),

then

\[ \text{span} \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r \} = \text{span} \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \]

if and only if

Every vector in \( S \) is a linear comb. of vectors in \( S' \)

and

Every vector in \( S' \) is a linear comb. of vectors in \( S \).
Example (book) Subspaces of $\mathbb{R}^3$.

$$A\hat{x} = \hat{0} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the soln. space.

Row reducing augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - 2y + 3z = 0$

Solution space is this plane through orig.

$$A\hat{x} = \hat{0}, \quad \begin{bmatrix} 1 & 5 & 7 & 0 \\ 2 & 4 & 2 & 0 \\ 3 & 2 & -5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 7 & 0 \\ 2 & 4 & 2 & 0 \\ 3 & 2 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x = 3z$, $y = 2z$, $z$ free

Soln. space is a line through origin

$x = 3t$, $y = -2t$, $z = t$. 
Example 15. - (Sect 5.2)

\[ \mathbf{v}_1 = (1,1,2), \quad \mathbf{v}_2 = (1,0,1), \quad \mathbf{v}_3 = (2,1,3) \]

Do not span \( \mathbb{R}^3 \).

If they do, then any vector \( \mathbf{v} = (b_1, b_2, b_3) \) in \( \mathbb{R}^3 \) will be a lin. comb. of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \).

Or

\[
K_1 \mathbf{v}_1 + K_2 \mathbf{v}_2 + K_3 \mathbf{v}_3 = (b_1, b_2, b_3)
\]

Or

\[
K_1 \begin{pmatrix}
1
1
2
\end{pmatrix} + K_2 \begin{pmatrix}
1
0
1
\end{pmatrix} + K_3 \begin{pmatrix}
2
1
3
\end{pmatrix} = \begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}
\]

Or

\[
K_1 + K_2 + 2K_3 = b_1 \\
K_1 + K_3 = b_2 \\
2K_1 + K_2 + 3K_3 = b_3
\]

Or

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

Or

\[
A \mathbf{k} = \mathbf{b} \quad (\ast)
\]
Now, \[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 3
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 2 \\
0 & -1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]
\[ \Rightarrow \det A = 0 \Rightarrow \text{Equiv. thm.} \]
Nonhomog. syst. (\#) is not consist. for some \( b \).

Example 9. (Sect. 4.2)

\( \hat{\mathbf{u}} = (1, 2, -1), \quad \hat{\mathbf{v}} = (6, 4, 2) \) in \( \mathbb{R}^3 \)
and \( \hat{\mathbf{w}} = (9, 2, 7), \quad \hat{\mathbf{w}}' = (4, 2, 7) \)
Prove that \( \hat{\mathbf{w}} \) is lin. comb. but \( \hat{\mathbf{w}}' \) is not

\( (9, 2, 7) = k_1 (1, 2, -1) + k_2 (6, 4, 2) \)

\[
\begin{align*}
k_1 + 6k_2 &= 9 \\
2k_1 + 4k_2 &= 2 \\
-k_1 + 2k_2 &= 7
\end{align*}
\]
\[
\Rightarrow \begin{bmatrix}
1 & 6 & 9 \\
2 & 4 & 2 \\
-1 & 2 & 7
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]
\[ k_1 = -3, \quad k_2 = 2. \]
For \( \tilde{\mathbf{w}} \), we have

\[
\begin{align*}
K_1 + 6K_2 &= 4 \\
2K_1 + 4K_2 &= 2 \\
-K_1 + 2K_2 &= 7
\end{align*}
\]

\[
\begin{bmatrix}
1 & 6 & 4 \\
2 & 4 & 2 \\
-1 & 2 & 7
\end{bmatrix}
\]

inconsistent!

Does it mean \( S = \{ \tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}} \} \) is lin. indep?

\[
\begin{aligned}
\text{Example 4.3} & \quad \text{(Sec. 4.3)} \\
\tilde{\mathbf{v}}_1 &= \begin{pmatrix}
\frac{1}{2} \\
\frac{1}{3}
\end{pmatrix}, \quad \tilde{\mathbf{v}}_2 &= \begin{pmatrix}
\frac{5}{6} \\
-1
\end{pmatrix}, \quad \tilde{\mathbf{v}}_3 &= \begin{pmatrix}
\frac{3}{2} \\
1
\end{pmatrix}
\end{aligned}
\]

Clearly,

\[
\tilde{\mathbf{v}}_1 + \tilde{\mathbf{v}}_2 = \begin{pmatrix}
\frac{6}{12} \\
\frac{4}{3}
\end{pmatrix} = 2 \begin{pmatrix}
\frac{3}{2} \\
1
\end{pmatrix} = 2 \tilde{\mathbf{v}}_3
\]

The set \( \{ \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3 \} \) is lin. dep.

Procedure to determine lin. dep. condition.

\[
K_1 \begin{pmatrix}
\frac{1}{2} \\
\frac{1}{3}
\end{pmatrix} + K_2 \begin{pmatrix}
\frac{5}{6} \\
-1
\end{pmatrix} + K_3 \begin{pmatrix}
\frac{3}{2} \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

equiv. to

\[
\begin{pmatrix}
1 & 5 & 3 \\
-2 & 6 & 2 \\
3 & -1 & 1
\end{pmatrix}
\begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 0 & \frac{1}{2} & 0 \\
0 & 1 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\Rightarrow
K_3 = \frac{1}{2}K_2
\]

\[
K_3 = -\frac{1}{2}K_2
\]

\[
K_3 = -\frac{1}{2}K_2
\]