Consider again the problem in the semi-infinite line

\[ \begin{align*}
U_t & = C^2 U_{xx}, \quad 0 < x < L, \ t > 0 \\
U(0,t) & = 0 \\
U(x,0) & = f(x), \quad U_t(x,0) = g(x)
\end{align*} \]

Direnchlet \[ (1) \]

IBVP \[ (2) \]

Soh. is given by

\[ U(x,t) = \begin{cases} 
\frac{1}{2} \left[ f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds, & x-ct > 0 \\
\frac{1}{2} \left[ f(x+ct) - f(ct-x) \right] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) \, ds, & x-ct < 0
\end{cases} \]

\[ (4) \]

Also, consider the problem in the infinite line

\[ \begin{align*}
U_t & = C^2 U_{xx}, \quad -\infty < x < \infty, \ t > 0 \\
U(x,0) & = \begin{cases} 
f(x), & x > 0 \\
-f(-x), & x < 0
\end{cases}, \quad U_t(x,0) = \begin{cases} 
g(x), & x > 0 \\
-g(x), & x < 0
\end{cases}
\end{align*} \]

Odd extension of initial conditions

\[ (6) \]

Theorem: The IBVPs (1)-(3) and (5)-(6) are equivalent over their common domains \[ 0 \leq x < \infty, \ t > 0. \]
Proof: The soln. of (5)-(6) using D'Alambert's formula is

\[ U(x,t) = \frac{1}{2} \left[ f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} G(x) \, dx \]

\[ U(x,t) = \begin{cases} 
\frac{1}{2} \left[ f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} G(x) \, dx, & x-ct > 0. \\
\frac{1}{2} \left[ f(x+ct) - f(ct-x) \right] + \frac{1}{2c} \left[ \int_{x-ct}^{0} G(x) \, dx + \int_{0}^{x+ct} G(x) \, dx \right] & (x-ct < 0) \quad \text{and} \quad (x+ct > 0) 
\end{cases} \]

Now,

\[ \int_{x-ct}^{0} G(x) \, dx = \int_{0}^{x-ct} G(-x) \, dx = -\int_{ct-x}^{0} G(u) \, du = \int_{ct-x}^{0} G(u) \, du \]

and

\[ \int_{0}^{x+ct} G(x) \, dx = \int_{0}^{x+ct} G(x) \, dx \]

Therefore,

\[ U(x,t) = \frac{1}{2} \left[ f(x+ct) - f(ct-x) \right] + \frac{1}{2c} \int_{ct-x}^{0} G(u) \, du \quad \text{when} \quad (x-ct < 0) \quad \text{and} \quad (x+ct > 0) \]

Therefore, the solns of (1)-(3) and (5)-(6) are identical in their common domains. 0 ≤ x < ∞, t > 0.
12.4.1 (More General)

\[
\begin{align*}
U_t &= c^2 U_{xx}, & x > 0, & t > 0 & \quad (1) \\
U(x,0) &= f(x), & x > 0 & \quad (2) \\
U_t(x,0) &= g(x), & x > 0 & \quad (3) \\
U(0,t) &= h(t), & t > 0 & \quad (4)
\end{align*}
\]

General solution of (1):

\[U(x,t) = F(x-ct) + G(x+ct)\] \hfill (6.1)

Using IC's is possible to determine \( F \) and \( G \) for \( x > 0 \).

In fact,

\[f(x) = U(x,0) = F(x) + G(x), \quad x > 0\] \hfill (5)

\[g(x) = U_t(x,0) = -cF'(x) + CG(x), \quad x > 0\] \hfill (6)

\[f'(x) = F'(x) + G'(x)\]

\[f'(x) + \frac{1}{c} g(x) = 2G'(x) \Rightarrow\]

\[G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(x) \, dx + K\]

From (5)

\[F(x) = f(x) - G(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(x) \, dx - K\]

Summarizing,

\[
\begin{align*}
F(x) &= \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(x) \, dx - K, & x > 0 \quad (5.2)
\end{align*}
\]

\[
\begin{align*}
G(x) &= \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(x) \, dx + K, & x > 0 \quad (5.3)
\end{align*}
\]
Therefore, the soln. of (1)-(4) is completely known when 

\[ x - ct > 0 \]

\[ U(x,t) = F(x - ct) + G(x + ct) = \frac{1}{2} \left[ f(x - ct) + f(x + ct) \right] + \frac{1}{2c} \int_{x - ct}^{x + ct} g(\tau) d\tau \]  

(7)

However, when 

\[ x - ct < 0 \]

\[ F(x - ct) \] is not defined.

To obtain the soln. in this region, we use (5.1) and (4).

\[ h(t) = U(0,t) = F(-ct) + G(ct) \]  

(8)

\[ F(-ct) \] is not defined. However, from (8)

\[ F(-ct) = h(t) - G(ct) \]

if \( z = -ct \) then, \( ct = -z \) and \( t = \frac{z}{c} \)

Thus,

\[ F(z) = h(-\frac{z}{c}) - G(-\frac{z}{c}), \quad z < 0 \]  

(9)

Eq. (4) represents the def. of \( F \) for negative numbers.

Therefore, the solution \( u(x,t) \) when \( x - ct < 0 \) can be written as

\[ U(x,t) = F(x - ct) + G(x + ct) = h \left( \frac{ct - x}{c} \right) - \frac{1}{2} \int_{0}^{ct-x} g(\tau) d\tau + \frac{1}{2c} \int_{ct-x}^{ct} g(\tau) d\tau + h \left( \frac{t-x}{c} \right) \]  

(10)

\[ U(x,t) = \frac{1}{2} \left[ f(x + ct) - f(x - ct) \right] + \frac{1}{2c} \int_{x - ct}^{x + ct} g(\tau) d\tau + h \left( \frac{t-x}{c} \right) \]  

(11)
Therefore, solution is given by

\[
U(x,t) = \begin{cases} 
\frac{1}{2} \left[ f(x-ct) + f(x+ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) \, d\xi, & x-ct > 0 \\
\frac{1}{2} \left[ f(x+ct) - f(ct-x) \right] + \frac{1}{2c} \int_{ct-x}^{ct} g(\xi) \, d\xi + h(t-x/c), & x-ct < 0
\end{cases}
\]
12.4.4 Same as 12.4.1 (general) but B.C.: \( U_x(0,t) = g(t) \).

\[
\begin{align*}
U_t &= kU_{xx}, \quad x > 0, \quad t > 0 \\
U(x,0) &= f(x), \quad x > 0 \\
U_t(x,0) &= g(x), \quad x > 0 \\
U_x(0,t) &= f(t), \quad t > 0
\end{align*}
\]

(1) Neumann B.C. at \( x = 0 \).

General soln: \( U(x,t) = F(x-ct) + G(x+ct) \). (5)

From IC's.

\[
\begin{align*}
F(x) &= \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(r) \, dr, \quad x > 0 \\
G(x) &= \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(r) \, dr, \quad x > 0
\end{align*}
\]

The constant \( k \) can be omitted because it will cancel when \( F \) and \( G \) are added.

Again (as in 12.4.1) the soln in region \( x - ct > 0 \) is given by

\[
U(x,t) = F(x-ct) + G(x+ct) = \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(r) \, dr
\]

On region: \( x - ct < 0 \), \( F(x-ct) \) is not defined.

We want to use B.C. (4). So we need

\[
U_x(x,t) = F'(x-ct) + G'(x+ct)
\]

and

\[
f(t) = U_x(0,t) = F'(-ct) + G'(ct), \quad t > 0
\]

\[
\Rightarrow F'(-ct) = f(t) - G'(ct)
\]
Calling \( z = -ct \) \( \Rightarrow t = -\frac{z}{c} \), and \( ct = -z \).

Therefore,
\[
F'(z) = J(-\frac{z}{c}) - G'(-z)
\]

To obtain \( F(z) \), we integrate
\[
\int_0^z F'(\tilde{z}) d\tilde{z} = \int_0^z J\left(-\frac{\tilde{z}}{c}\right) d\tilde{z} - \int_0^z G'(\tilde{z}) d\tilde{z}
\]

First notice,
\[
\int_0^z J\left(-\frac{\tilde{z}}{c}\right) d\tilde{z} = \int_{\tilde{u} = -\frac{z}{c}}^{\tilde{u} = -\frac{z}{c}} J(\tilde{u})\left(-\frac{c}{z}\right) d\tilde{u} = -c \int_0^{-\frac{z}{c}} J(\tilde{u}) d\tilde{u}
\]

Therefore,
\[
F(z) - F(0) = -c \int_0^{-\frac{z}{c}} J(\tilde{u}) d\tilde{u} + G(-z) - G(0), \quad z < 0.
\]

Using the definition of \( G \) and replacing \( z = x - ct \).
\[
F(x - ct) = G(ct - x) - c \int_0^{\frac{t-x}{c}} J(\tilde{u}) d\tilde{u} + H
\]

Then,
\[
U(x, t) = F(x - ct) + G(x + ct) = G(x + ct) + G(ct - x) - c \int_0^{\frac{t-x}{c}} J(\tilde{u}) d\tilde{u} + H.
\]

When \( x - ct < 0 \).
Writing the solution in terms of \( f \) and \( g \):

\[
U(x,t) = \begin{cases} 
\frac{1}{2} \left[ f(x-ct) + f(x+ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(x) \, dx, & x-ct > 0 \\
\frac{1}{2} \left[ f(x-ct) + f(ct-x) \right] + \frac{1}{2c} \int_{0}^{ct-x} g(x) \, dx + \frac{1}{2c} \int_{ct-x}^{x+ct} g(x) \, dx \\
-\int_{0}^{b-x_0} f(\hat{u}) \, d\hat{u} + H, & x-ct < 0.
\end{cases}
\]  

(5.1)

Remark: Continuity at \((0,0)\) implies \( H = 0 \).

b) If \( U_x(0,t) = 0 \), the solution of IBVP (1)-(4) can be obtained by extending the initial position \( f \) and the initial velocity \( g \) as even functions around \( x = 0 \).

Proof:

Define \( \hat{f}(x) = \begin{cases} f(x), & x > 0 \\
-f(-x), & x < 0 \end{cases} \)

Even extension of \( f(x) \).

And consider the IVP for \( \hat{U} \):

\[
\begin{align*}
\hat{U}_t &= \hat{C}^2 \hat{U}_{xx}, & -\infty < x < \infty, \ t > 0 \\
\hat{U}(x,0) &= \hat{f}(x), & \hat{U}_t(x,0) = \hat{g}(x).
\end{align*}
\]
It's solution is given by

\[ U(x,t) = \frac{1}{2c} \left[ f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\tilde{x}) d\tilde{x} \]

in terms of \( f \) and \( g \).

\[ U(x,t) = \begin{cases} 
\frac{1}{2} \left[ f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\tilde{x}) d\tilde{x}, & x-ct > 0 \\
\frac{1}{2} \left[ f(x+ct) + f(0) \right] + \frac{1}{2c} \int_{x-ct}^{0} g(-\tilde{x}) d\tilde{x} + \frac{1}{2c} \int_{0}^{x+ct} g(\tilde{x}) d\tilde{x}, & x-ct < 0
\end{cases} \]

This is identical to (5.1) when \( f(t) \equiv 0 \).

Discuss practical problems for both cases:

\[ f(x) = \begin{cases} 
1, & 0 < x < 5 \\
0, & \text{otherwise}
\end{cases} \quad g(x) \equiv 0. \]

\[ u = -x \]

\[ \int_{0}^{x} g(u) (-du) = - \int_{ct-x}^{0} g(u) du \]
METHOD OF CHARACTERISTIC FOR
THE WAVE EQUATION IN THE FINITE LINE.

\[ u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \ t > 0 \]

\[ u(0,t) = h(t) \]
\[ u(L,t) = g(t) \] \{ BC's. DIReCTLIET TYPE \}

\[ u(x,0) = f(x) \] \{ IC's. \}
\[ u_t (x,0) = g(x) \]

\[ t \]
\[ u=h(t) \]
\[ u=g(t) \]
\[ t=L/c \]
\[ x+ct = 2L \]
General Soln:

\[ U(x,t) = F(x-ct) + G(x+ct) \]

Where

\[ F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\xi) d\xi, \quad 0 < x < L \]

\[ G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\xi) d\xi, \quad 0 < x < L \]

In region I: \( 0 < x-ct < L \), \( 0 < x+ct < L \)

\[ U(x,t) = F(x-ct) + G(x+ct) \]

\[ \Rightarrow U(x,t) = \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi \]

In region II: \( -L < x-ct < 0 \), \( 0 < x+ct < L \)

\( F(x-ct) \) is not defined

Using B.C. at \( x = 0 \)

\[ h(t) = U(0,t) = F(-ct) + G(ct) \]

\( \Rightarrow F(-ct) = h(t) - G(ct) \)

Calling \( z = -ct \) \( \Rightarrow t = -\frac{z}{c} \)

\( \Rightarrow F_1(z) = h\left(-\frac{z}{c}\right) - G\left(-\frac{z}{c}\right) \)

\( \Rightarrow F_1(x-ct) = h\left(\frac{ct-x}{c}\right) - G\left(\frac{ct-x}{c}\right) \)
In Region II:
The solution is given by
\[ U(x, t) = F(x - ct) + G(x + ct) = h(t - \frac{x}{c}) - G(ct - x) + G(x + ct) \]
\[ = \frac{1}{2} \left[ f(x + ct) - f(ct - x) \right] + \frac{1}{2c} \int_0^{x + ct} g(2v) dv - \frac{1}{2c} \int_0^{ct - x} g(2v) dv \]

or
\[ U(x, t) = \frac{1}{2} \left[ f(x + ct) - f(ct - x) \right] + \frac{1}{2c} \int_{ct - x}^{x + ct} g(2v) dv + h(t - \frac{x}{c}) \]

In Region III:
\( 0 < x - ct < L, \quad L < x + ct < 2L \)
\( G(x + ct) \) is not defined.

Using BC at \( x = L \)
\[ J(t) = U(L, t) = F(L - ct) + \overbrace{G(L + ct)}^{t \geq \frac{L}{c}} \]
\[ \Rightarrow G_1(L + ct) = J(t) - F(L - ct) \]

Calling \( z = L + ct \) \( t = \frac{z - L}{c} \) \( \Rightarrow ct = \frac{z - L}{c} \) \( \Rightarrow L - ct = 2L - 2z \)
\[ G_1(z) = J \left( \frac{z - L}{c} \right) - F(2L - z) \]

If \( L < z < 2L \) \( -2L < z - L < L \) \( \Rightarrow 0 < 2L - z < L \)
\[ \Rightarrow G_1(x + ct) = J \left( \frac{x + ct - L}{c} \right) - F(2L - x - ct) \]

And the solution
\[ U(x, t) = F(x - ct) + G(x + ct) = \frac{1}{2} \left[ f(x + ct) - f(2L - x - ct) \right] + \]
\[ - \frac{1}{2c} \int_0^{x + ct} g(2v) dv + \frac{1}{2c} \int_0^{2L - x - ct} g(2v) dv + f(t + \frac{x}{c}) \]
or In region 3:

\[ U(x,t) = \frac{1}{\beta} \left[ f(x-ct) - f(2L-x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{2L-x-ct} g(z) \, dz \]

In region II: \(-L < x-ct < 0, \quad L < x+ct < 2L\)

\[ U(x,t) = F(x-ct) + G(x+ct) = F_1(x-ct) + G_1(x+ct) \]

\[ = h \left( \frac{ct-x}{c} \right) - G(2t-x) + F \left( \frac{x+ct-L}{c} \right) - F(2L-x-ct). \]

- \( F_1 \) was obtained in terms of original \( G \).

\( G_1 \) _____________________ F.

In region V: \(-2L < x-ct < -L, \quad L < x+ct < 2L\)

\[ U(x,t) = \frac{1}{2} \left[ \tilde{F}(x-ct) + \tilde{G}(x+ct) \right] \]

\[ h(t) = U(t,ct) = F_2(-ct) + G(ct) \Rightarrow F_2(-ct) = h(t) - G_1(ct) \]

\[ \Rightarrow F_2(z) = h \left( \frac{z}{c} \right) - G \left( \frac{z}{c} \right) \text{ on } \Omega \]

If \(-2L < z < -L \Rightarrow L < -z < 2L\)

\[ \Rightarrow F_2(x-ct) = h \left( \frac{ct-x}{c} \right) - G_1(ct-x) = \]

\[ = h \left( \frac{x}{c} \right) - F \left( \frac{x+ct-L}{c} \right) + F \left( 2L + x - ct \right) \]
In region III: \(-L < x - ct < 0\), \(2L < x + ct < 3L\)

\[ U(x,t) = \frac{\psi}{F_1(x-ct)} + G_2(x+ct) \]

B.C.

\[ J(t) = U(L,t) = F_1(L-ct) + G_2(L+ct) \]

\( \Rightarrow \quad G_2(L+ct) = J(t) - F_1(L-ct) \)

\[ z = L+ct \]
\[ G_2(z) = J\left(\frac{z-L}{c}\right) - F_1\left(\frac{z-L-t}{c}\right) \]

\[ ct = \frac{t-L}{c} \quad -ct = \frac{L-t}{c} \]

If \(2L < z < 3L\) \(\Rightarrow\) \(-3L < z < -2L\) \(\Rightarrow\) 

\[ z < -L \quad z > L \]

\[ U(x,t) = F(x-ct) + G_2(x+ct) = \]

\[ = \ h\left(\frac{ct-x}{c}\right) - G(ct-x) + G_2(x+ct) = \]

\[ = \ h\left(\frac{t-xc}{c}\right) - G(ct-x) + J\left(\frac{x+ct-L}{c}\right) \]

\[ - F_1\left(2L-(x+ct)\right) \]

\[ = \ h\left(\frac{t-xc}{c}\right) - G(ct-x) + J\left(\frac{x+ct-L}{c}\right) \]

\[ - h\left(\frac{2L+xc}{c}\right) - G\left(x+ct-2L\right) \]

In region VII: \(-2L < x - ct < -L\), \(2L < x + ct < 3L\)

\[ U(x,t) = F_2(x-ct) + G_2(x+ct) \]
\[ u(t,x) = f(x) = \begin{cases} 1, & 0 < x < 5 \\ 0, & \text{otherwise} \end{cases} \]

\[ u_{t}(x,0) = g(x) = 0 \]

\[ u_{tt} = c^2 u_{xx} \]

\[ u(0,t) = 0 \]

\[ u(10,t) = 0 \]
What Kind of Extension?  # 12.5.4

1. Odd extension of I.C. around $x = L$.
2. Even extension of I.C. around $x = 0$.
3. Periodic extension of the function defined in $(0, 4L]$ with period $4L$.

To identify a pattern, find the soln. inside the interval $[0, L]$ by extending the I.C. using both BC's to the interval $[0, 4L]$. 