3.4 Term by Term Differentiation of Fourier Series.

What is the Fourier Cosine Series of \( f(x) \equiv 1 \) over \([0, L] \)?

\[
1 \sim = \sum_{n=0}^{\infty} A_n \cos \left( \frac{n\pi}{L} x \right)
\]

\[
A_0 = \frac{1}{L} \int_0^L 1 \, dx = 1
\]

\[
A_n = \frac{2}{L} \int_0^L \cos \left( \frac{n\pi}{L} x \right) \, dx = 0, \quad n=1, 2, \ldots
\]

Then,

\[
1 = 1 \checkmark
\]

On the other hand, Fourier Sine Series of \( f(x) = x \), on \([0, L] \).

\[
x = \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sin \left( \frac{n\pi}{L} x \right), \quad 0 \leq x \leq L, \quad \text{Jump at } x = L.
\]

Differentiating term by term,

\[
\frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \pi \cos \left( \frac{n\pi}{L} x \right)
\]

\[
= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \left( \frac{n\pi}{L} x \right) \neq 1.
\]
Thus.

If
1) Fourier Series of \( f(x) \) over \([-L,L]\) is continuous.
2) \( f(x) \) is p.w.s.

Then, F.S. Can be differentiated term by term.

Alternative

If
1) \( f'(x) \) is piecewise smooth on \([-L,L]\)
2) \( f(x) \) is continuous on \([-L,L]\)
3) \( f(-L) = f(L) \).

Then, F.S. Can be differentiated term by term.

Apply to

Fourier Cosine Series

Prop. 1.
If
1) \( f'(x) \) is p.w.s. on \([0,L]\)
2) \( f(x) \) is cont. on \([0,L]\)

Then, F. Can be diff. t. by t.

Example.

\( f(x) = x \), on \([0,L]\)

\[
X = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd only}} \frac{1}{n^2} \cos \left( \frac{n \pi x}{L} \right)
\]

Diff. term by term

\[
1 = \frac{4}{\pi} \sum_{n \text{ odd only}} \frac{1}{n} \sin \left( \frac{n \pi x}{L} \right)
\]
Prop. 2. — If 
1) \( f(x) \) is p.w.s. on \([0, L]\) .
2) \( f(x) \) is conts. on \([0, L]\) .
3) \( f(0) = 0, \ f(L) = 0 \).

Then, Fourier sine series can be differentiated term by term.

Proof. — Since \( f(x) \) is conts. on \([0, L]\)

\[
f(x) \sim \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \quad (3.1)
\]

Where \( B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi y}{L} \right) \, dx \).

Also, since \( f'(x) \) is p.w.s. on \([0, L]\).

\[
f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right) \quad (3.2)
\]

Where \( A_0 = \frac{1}{L} \int_{0}^{L} f(x) \, dx \)

\[
A_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx .
\]
If (3.1) were differentiated term by term
\[ f'(x) \sim \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right) B_n \cos \left( \frac{n\pi}{L} x \right) \]

and comparing with (3.2), it should be verified that
\[ A_0 = 0, \quad A_n = \frac{n\pi}{L} B_n. \quad (4.1) \]

Equations (4.1) actually hold. In fact,
\[ A_0 = \frac{1}{L} \int_{0}^{L} f(x) \, dx = \frac{1}{L} \left[ f(L) - f(0) \right] = 0. \quad (f(L) = f(0)) \]

Also,
\[ A_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{n\pi}{L} x \right) \, dx \rightarrow \int_{0}^{L} f(x) \cos \left( \frac{n\pi}{L} x \right) \, dx = \frac{2}{L} \left[ f(L) \cos \left( \frac{n\pi}{L} L \right) - f(0) \right] + \frac{n\pi}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi}{L} x \right) \, dx \]
\[ = \frac{2}{L} \left[ (-1)^n f(L) - f(0) \right] + \frac{n\pi}{L} B_n = \frac{n\pi}{L} B_n. \quad \checkmark \]

We have actually proved that

Prop 3: If 1) \( f(x) \) is p.w.s on \([0, L] \)
2) \( f(x) \) is cont. on \([0, L] \)
then, if \( f(x) \sim \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi}{L} x \right) \)
\[ \Rightarrow f'(x) \sim \frac{1}{L} \left[ f(L) - f(0) \right] + \sum_{n=1}^{\infty} \left[ \frac{n\pi}{L} B_n + \frac{2}{L} \left( (-1)^n f(L) - f(0) \right) \right] \cos \left( \frac{n\pi}{L} x \right) \]