Derivation of the Heat Equation

Unsteady heat conduction in a rod laterally perfectly insulated


\[
\frac{\partial}{\partial t} \left( e(x,t) \right) = \frac{\partial}{\partial x} \left( \phi(x,t) \right) + Q(x,t)
\]

where \( A \) is area of cross-section.
Energy density $e(x,t)$ in terms of temperature $V(x,t)$.

Let's introduce

$V(x,t)$: Temperature

$\rho(x)$: Density (Mass)

$C(x)$: Specific heat

Then, their dimensions are given by

$[V] = T$, $[\rho] = \frac{M}{L^3}$, $[C] = \frac{E}{M \cdot T}$

Therefore,

$e(x,t) = C(x) \rho(x) V(x,t)$

In fact,

$[e] = [C][\rho][V] = \frac{E}{\sqrt{M}} \frac{M}{L^3} T = \frac{E}{L^3}$

Then (2) is transformed into

$$\frac{\partial}{\partial t} (C(x)\rho(x) V(x,t)) = -\frac{\partial \phi}{\partial x} (x,t) + Q(x,t)$$

or

$$C(x)\rho(x) \frac{\partial V}{\partial t} (x,t) = -\frac{\partial \phi}{\partial x} (x,t) + Q(x,t)$$

(3)
Fourier Law's:

\[ C(x) \rho(x) \frac{\partial \mathbf{V}}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t). \]

\( C(x), \rho(x), Q(x,t) \): Known
\( \mathbf{V}(x,t), \phi(x,t) \): unknown. \[ \begin{cases} \phi > 0 \Rightarrow \text{Energy flows right} \\ \phi < 0 \Rightarrow \text{Energy flows left} \end{cases} \]

Fourier's law establishes a relationship between these two unknowns.

This law should include familiar qualitative properties as

1) No flow if \( \mathbf{V}(x,t) \) constant.
   \( \text{(Temperature)} \)

2) Heat energy flows from hotter regions to colder regions.

3) Greater temperature differences implies greater flow of heat energy.

4) Flow of heat energy depends on the material.
Therefore, a good "law" or relationship is given by

\[ \phi(x,t) = -K_0(x) \frac{\partial V}{\partial x}(x,t) \quad (4) \]

Verify that conditions (1) - (4) holds

Substituting (4) into (3), leads to

\[ C(x) \rho(x) \frac{\partial V}{\partial t}(x,t) = \frac{\partial}{\partial x} \left( K_0(x) \frac{\partial V}{\partial x}(x,t) \right) + \frac{Q(x,t)}{C(x) \rho(x)} \]

or

\[ \frac{\partial V}{\partial t}(x,t) = \frac{1}{C_0 \rho_0} \frac{\partial}{\partial x} \left( K_0(x) \frac{\partial V}{\partial x}(x,t) \right) + \frac{Q(x,t)}{C_0 \rho_0} \]

only one unknown \( V(x,t) \), the temperature.

If \( K_0(x) \equiv K_0 \) constant, \( C(x) \equiv C_0 \), and \( \rho(x) \equiv \rho \) constant.

then, we arrive to

\[ \frac{\partial V}{\partial t} = \kappa \frac{\partial^2 V}{\partial x^2} + f(x,t) \]

where \( \kappa \equiv \frac{K_0}{C_0 \rho_0} \) thermal diffusivity.

and \( f(x,t) = \frac{Q(x,t)}{C_0 \rho_0} \).
Boundary Conditions:

**Dirichlet:** \( \mathbf{U}(0,t) = A(t) \quad \mathbf{U}(L,t) = B(t) \) or \( \mathbf{U}(0,t) = C(t) \quad \mathbf{U}(L,t) = D(t) \) Prescribed Temp.

**Neumann:** \( \frac{\partial \mathbf{U}}{\partial x}(0,t) = C(t) \), \( \frac{\partial \mathbf{U}}{\partial x}(L,t) = D(t) \) Prescribed flow

**Robin:** \( \frac{\partial \mathbf{U}}{\partial x}(0,t) + H \mathbf{U}(0,t) = T_0(t) \). Neat. law of cooling.

**Dirichlet BVP:**

\[
\begin{cases}
  \mathbf{U}_t + k \mathbf{U}_{xx} = 0, & 0 < x < L, \ t > 0 \\
  \mathbf{U}(0,t) = A, \quad \mathbf{U}(L,t) = B \\
  \mathbf{U}(x,0) = f(x)
\end{cases}
\]

For \( f \) p.w.s.

the Soln. for this BVP (Sep. Variables)

\[
\mathbf{U}(x,t) = A + \frac{B-A}{L} x + \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{L} x \right) e^{-k \left( \frac{n\pi}{L} \right)^2 t}
\]

Where

\[
A_n = \frac{2}{L} \int_0^L \left[ f(x) - \left( A + \frac{B-A}{L} x \right) \right] \sin \left( \frac{n\pi}{L} x \right) \ dx.
\]
Clearly, \( U(x,t) \to A + \frac{B-A}{L} x \quad \text{Steady State} \quad t \to \infty \)

All initial effects have disappeared.

This solution can also be obtained from the equilibrium BVP

\[
\frac{\partial^2 U}{\partial x^2} = 0 \\
\frac{\partial U}{\partial t} = 0 \\
U(0) = A, \quad U(L) = B
\]

Integrating twice:

\[
U(x) = C_1 x + C_2
\]

\( A = U(0) = C_2 \quad \text{and} \quad U(L) = C_1 L + A \)

\[
C_1 = \frac{B-A}{L}
\]

Thus, solution:

\[
U(x) = \frac{B-A}{L} x + A
\]