2.13 Existence and Uniqueness

Case 1: Nonhomogeneous Dirichlet Problem.

\[
\begin{align*}
  U''(x) &= f(x), & 0 < x < 1 & (1.1) \\
  U(0) &= \alpha, & U(1) &= \beta & (1.2)
\end{align*}
\]

Solution: Assuming \( G'(x) = \int f(x) \) then, \( U(x) = G(x) + C_1 x + C_2 \) (integrating (1.1) twice)

Using the BCs

\[
\begin{align*}
  \alpha &= U(0) = G(0) + C_2 \\
  \beta &= U(1) = G(1) + C_1 + \alpha - G(0)
\end{align*}
\]

\[ \Rightarrow C_1 = \beta - G(1) - \alpha + G(0) \]

And as a consequence,

\[
U(x) = G(x) + \left[ \beta - \alpha + G(0) - G(1) \right] x + \alpha - G(0). \tag{1.3}
\]

Is this solution unique?

Consider two possible solutions: \( U_1(x) \) and \( U_2(x) \) then, \( \omega(x) = U_1(x) - U_2(x) \) satisfies

\[
\begin{align*}
  \omega''(x) &= 0, & 0 < x < 1 & (1.4) \\
  \omega(0) &= 0, & \omega(1) &= 0 & (1.5)
\end{align*}
\]

Remark: From Physical physics, (1.1) - (1.2) always has a solution because an equilibrium can be reached whenever at least in one end the flux is not prescribed.
Temperature is fixed at the ends but flux is not

Any excess of thermal energy generated inside the rod will be going out at the ends so temp at ends remains unchanged.

Same if heat is absorbed inside bar (Sink).

In this case, the flow will be directed inside the bar.
It can be shown that the only solution of this BVP is the trivial solution: \( w(x) = 0 \) for all \( x \in [0,1] \).

This is immediately seen by direct integration.

Then, \( u_1(x) = u_2(x) \)

and BVP (1.1) - (1.2) has a unique soln.

Compare this results for the homogeneous and nonhomogeneous BVPs with the following theorem for linear systems of algebraic equations for a matrix \( A \times n \times n \).

\[ A \hat{x} = \hat{b} \text{ has only the trivial solution } \iff \quad \text{Nonhomogeneous system } A \hat{x} = \hat{b} \text{ has a unique solution for any } \hat{b}. \]

Consider

Case 2. - Nonhomogeneous Neumann Problem

\[
\begin{align*}
\left\{ \begin{array}{l}
U''(x) = f(x), \quad 0 < x < 1 \\
U'(0) = \sigma_0, \quad U'(1) = \sigma_1
\end{array} \right. \\
(2.1)
\end{align*}
\]

Discuss the existence of solutions from physical points first.

Consider the two cases: \( \sigma_0, \sigma_1 \neq 0 \) and \( \sigma_1 = \sigma_0 = 0 \).
By integrating,

\[ U(x) = G(x) + C_1 x + C_2, \quad \text{where} \quad G'(x) = \int f(x) \, dx \]

Using BCs:

\[ U'(x) = G'(x) + C_1, \quad \text{then} \]

\[ \sigma_0 = U'(0) = G'(0) + C_1 \Rightarrow C_1 = \sigma_0 - G'(0) \]

\[ \sigma_1 = U'(1) = G'(1) + C_1. \Rightarrow C_1 = \sigma_1 - G'(1) \]

So the only possibility for existence of solutions is that

\[ G'(1) - G'(0) = \sigma_1 - \sigma_0. \quad \text{(3.1)} \]

Which is equivalent to

\[ \int_0^1 f(x) \, dx = \sigma_1 - \sigma_0 \quad \text{(3.2)} \]

Physical Interpretation?

Since

\[ U'(1) - U'(0) = \int_0^1 U''(x) \, dx = \int_0^1 f(x) \, dx, \]

by integration of (2.1) and using (2.2)

Condition (3.2) is called a compatibility condition.

In general, (3.2) is not satisfied because \( \sigma_1, \sigma_0 \) and \( f(x) \) are arbitrary and independent of each other.
In the special case that (3.2) is verified, the Soln. is not unique because $C_2$ is undetermined. In fact,

$$U(x) = G(x) + (G'(0) - \sigma_0)x + C_2$$  \hspace{1cm} (4.1)

What can you say about the corresponding homogeneous problem

$$
\begin{cases}
U''(x) = 0, & 0 < x < 1 \\
U'(0) = 0, & U'(1) = 0.
\end{cases}
$$  \hspace{1cm} (4.2)

Compare with the relationship between the linear system of algebraic equations with singular matrix $A$,

$$A\hat{x} = \hat{b} \text{ and } A\hat{x} = \hat{0}.$$  \hspace{1cm} (4.3)

How can the unknown constant $C_2$ be calculated from I.C.s of the original heat cond. BVP?
Special case: \( \sigma_1 = \sigma_2 = 0 \) and \( f(x) \equiv 0 \).

\[
\begin{aligned}
\begin{cases}
U''(x) &= 0 \\
U'(0) &= 0, \quad U'(1) = 0
\end{cases}
\end{aligned}
\]

\[ U(x) \equiv C_2 \quad \text{Equilibrium} \quad \text{Soln.} \quad \text{Infinitely many.} \]

Now, \( C_2 \) can be calculated from ICs of original heat conduction problem.

In fact, if soln. of PDE: \( \bar{U}(x,t) \)

and there is an equilibrium solution

\[
\frac{d}{dt} E(t) = \frac{d}{dt} \int_0^1 c_p \bar{U}(x,t) \, dx = 0
\]

\( \Rightarrow \quad E(\infty) = E(0) \)

or

\[
\int_0^1 c_p \bar{U}(x,\infty) \, dx = \int_0^1 c_p \bar{U}(x,0) \, dx
\]

or

\[
c_p \int_0^1 U(x) \, dx = c_p \int_0^1 U_0(x) \, dx
\]

\( \Rightarrow \quad C_2 = \int_0^1 U_0(x) \, dx. \quad \text{(AVG. Temp in \( \bar{U}(x,t) \))} \)
Determination of $C_2$ in (4.1). General case

We go back to the conservation of energy equation:

$$\frac{dE}{dt}(t) = \frac{d}{dt} \left[ \int_0^1 C \rho \bar{u}(x,t) dx \right] = \phi(0,t) \frac{\bar{u} = \sigma_0}{\bar{u} = \sigma_t} + A \int_0^1 F(x) dx$$

Where

$$-kF(x) = f(x)$$

Now, if there is equilibrium, then

$$\frac{dE}{dt}(t) = 0 \Rightarrow E(t) \equiv \text{constant}$$

$$\Rightarrow \sigma_1 - \sigma_0 = -\frac{1}{ko} \int_0^1 F(x) dx = \int_0^1 f(x) dx$$

Same as (3.2)

But also,

$$E(t_0) = \int_0^1 \rho \bar{u}(x,t_0) dx = \int_0^1 \rho \bar{u}(x,0) dx = \int_0^1 U_0(x) dx$$

or

$$\int_0^1 U(x) dx = \int_0^1 U_0(x) dx$$

or

$$\int_0^1 \left[ G(x) + (G(0) - \sigma_0) x \right] + C_2^2 dx = \int_0^1 U_0(x) dx$$

$$\Rightarrow C_2 = \int_0^1 U_0(x) dx - \int_0^1 \left[ G(x) + (G(0) - \sigma_0) x \right] dx$$

(5.1)
Consider the FD approx of (2.1)-(2.2).

Using centered difference at the two ends.

\[
A \begin{bmatrix}
-1 & h & 0 & \cdots & 0 \\
1 & -2 & 1 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & -1 \\
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\vdots \\
\hat{u}_m \\
\hat{u}_{m+1} \\
\end{bmatrix}
= 
\begin{bmatrix}
\hat{u}_0 + \frac{h}{2} f(x_0) \\
\hat{u}_1 \\
\vdots \\
\hat{u}_m \\
\hat{u}_{m+1} - \frac{h}{2} f(x_{m+1}) \\
\end{bmatrix}
\]

Singular matrix !!

In fact, \( A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \).

It can be shown (Homework 2) that this system has infinitely many solutions if Condition

\[
\frac{h}{2} f(x_0) + h \sum_{i=1}^{m} f(x_i) + \frac{h}{2} f(x_{m+1}) = 0
\]

which is trapezoidal rule approximation of (3.2).