Green's Formula (1-D)

\[(uv')' = uu' + uv''\]

If \(u'\) and \(v'\) are continuous in \([a, b]\) and \(u, v \in C^2(a, b)\), then,

\[
\int_a^b (uv')' \, dx = \int_a^b (u'v') \, dx + \int_a^b uv'' \, dx
\]

or

\[
uv'|_a^b = \int_a^b (u'v') \, dx + \int_a^b uv'' \, dx
\]

Now,

\[(uv')' - (u'v)' = uv'' - u''v\]

\[
\Rightarrow \int_a^b (uv')' \, dx - \int_a^b (u'v')' \, dx = \int_a^b (uv'') \, dx - \int_a^b (u''v) \, dx
\]

\[
\Rightarrow \int_a^b \left[ u(x)v''(x) - v(x)u''(x) \right] \, dx = \left[ u(x)v(x) - v(x)u'(x) \right]_a^b
\]
BVP with Nonhomogeneous BCs. Green's Functions.

\[ \begin{cases} -U''(x) = f(x), \\ U(0) = \alpha, \quad U(L) = \beta \end{cases} \quad (1) \]

Using Green's formula for $U$ as in (1) and the Green's function $G(x; x_0)$ associated to (1)-(2), we are led to

\[ \int_0^L \left[ u(x) \frac{d^2}{dx^2} (x; x_0) - G(x; x_0) u(x) \right] dx = \left| \left. \left( u(x) \frac{du}{dx} (x; x_0) - G(x; x_0) \frac{du}{dx} (x) \right) \right|_0^L \]

or

\[ \int_0^L u(x) \delta(x-x_0) dx = \int_0^L G(x; x_0) (-f(x)) dx \]

or

\[ U(x_0) = \int_0^L G(x; x_0) f(x) dx - \beta \frac{du}{dx} (L; x_0) + \alpha \frac{du}{dx} (0; x_0) \]

From the symmetry of $G$, $G(x; x_0) = G(x_0; x)$ and $\frac{du}{dx} (x; x_0) = \frac{du}{dx} (x_0; x)$.

\[ \text{In fact} \quad G(x_0; x) = \begin{cases} (1- \frac{x}{L}) x_0, & x_0 \leq x \\ (1- \frac{x}{L}) x, & x > x_0 \end{cases} = G(x; x_0) \]
Therefore,
\[
U(x_0) = \int_0^L G(x_0; x) f(x) \, dx - \beta \frac{dg}{dx} (x_0; L) + \alpha \frac{dg}{dx} (x_0)
\]

Reversing roles of \(x\) and \(x_0\)
\[
U(x) = \int_0^L G(x; x_0) f(x_0) \, dx_0 - \beta \frac{dg}{dx_0} (x; L) + \alpha \frac{dg}{dx_0} (x; 0) \quad (*)
\]

From the definition
\[
G(x; x_0) = \begin{cases} 
(1 - \frac{x_0}{L}) x, & x \leq x_0 \\
(1 - \frac{L}{x_0}) x_0, & x > x_0
\end{cases} \quad (**) 
\]

we obtain
\[
\frac{dg}{dx_0} (x; x_0) = \begin{cases} 
- \frac{x}{L}, & x \leq x_0 \\
1 - \frac{x}{L}, & x > x_0
\end{cases}
\]

In particular, \(\frac{dg}{dx_0} (L; x_0) = -\frac{x}{L}\) and \(\frac{dg}{dx_0} (0; x_0) = 1 + \frac{x}{L}\)

Substitution into (*) leads to
\[
U(x) = \int_0^L G(x; x_0) f(x_0) \, dx_0 + \beta \frac{x}{L} + \alpha \left(1 - \frac{x}{L}\right) \quad (**.3)
\]
Using the definition of $G(x)$, the solution $U(x)$ given by (4.3) can be written as

$$U(x) = \alpha \left(1 - \frac{x}{L}\right) + \int_{x_0}^{x} \left(1 - \frac{x}{L}\right) x_0 f(x_0) \, dx_0 + \int_{x_0}^{L} \left(1 - \frac{x_0}{L}\right) x f(x_0) \, dx_0 + \beta \frac{x}{L}$$

or

$$U(x) = \alpha \left(1 - \frac{x}{L}\right) + \left(1 - \frac{x}{L}\right) \int_{0}^{x} x_0 f(x_0) \, dx_0 + \int_{x}^{L} \left(1 - \frac{x_0}{L}\right) f(x_0) \, dx_0 + \beta \frac{x}{L} \quad (4.4)$$

In our textbook, $G(x; x_0)$ is the Green's function for

$$U''(x) = f(x), \quad U(0) = \alpha, \quad U(L) = \beta, \quad (\text{notice the change in sign})$$

Therefore,

$$G(x; x_0) = \begin{cases} \left(\frac{x_0}{L} - 1\right) x, & x \leq x_0 \\ \left(\frac{x}{L} - 1\right) x_0, & x > x_0 \end{cases} \quad (4.5)$$

It satisfies:

$$\frac{d^2 G}{dx^2}(x; x_0) = \delta(x - x_0), \quad G(0; x_0) = 0, \quad G(L; x_0) = 0$$
From Green's formula

\[ U(x_0) = \int_0^L G(x;x_0) f(x_0) \, dx + U(1) \frac{dG}{dx}(L;x_0) - U(0) \frac{dG}{dx}(0;x_0) \]

Using symmetry and reversing the roles of \( x \) and \( x_0 \).

\[ U(x) = \int_0^L G(x;x_0) f(x_0) \, dx_0 + \beta \frac{ds}{dx_0}(x;0^*) - \alpha \frac{dG}{dx_0}(x;0^*) \]

or

\[ \begin{cases} \[ U(x) = \alpha \left( 1 - \frac{x}{L} \right) + \int_0^L G(x;x_0) f(x_0) \, dx_0 + \beta \frac{x}{L} \] & \text{(same as (\ref{eq:3}) but } G(x;x_0) \text{ is the negative of the previous } G. \text{)} \end{cases} \]

Using the definition (\ref{eq:5}) for \( G(x;x_0) \)

\[ U(x) = \alpha \left( 1 - \frac{x}{L} \right) + \left( \frac{x}{L} - 1 \right) \int_0^x f(x_0) \, dx_0 + x \int_x^L \left( \frac{x_0}{L} - 1 \right) f(x_0) \, dx_0 + \beta \frac{x}{L} \]  \text{(same as (\ref{eq:3}) but } G(x;x_0) \text{ is the negative of the previous } G. \text{)}