CHAPTER 9  Diffusion Equations and Parabolic Problems.

Consider IBVP:

\[
\begin{align*}
U_t &= \ddot{U} + U_{xx}, \quad 0 < x < 1, \quad 0 < t. \\
U(x,0) &= \eta(x) \\
U(0,t) &= \varphi(t), \quad t > 0 \\
U(1,t) &= \psi(t), \quad t > 0
\end{align*}
\]

(1) (2) (3) (4)

Discretization: Grid

\[X_i = \Delta X, \quad i = 0, \ldots, N+1\]

\[t_n = n \Delta t, \quad n = 0, \ldots, M\]

\[h = \Delta X, \quad \Delta = \Delta t\]

\[U_i^n = U(x_i,t_n)\]

**FTCS numerical method**

(i) Approx. of \((U_t)_i^n, \) (forward difference)

\[U_i^{n+1} = U_i^n + \Delta (U_t)_i^n + \frac{\Delta^2}{2} (U_{tt})_i^n + \theta \]

\[\Rightarrow \quad (U_t)_i^n = \frac{U_i^{n+1} - U_i^n}{\Delta t} - \frac{\Delta^2}{2} (U_{tt})_i^n + \theta \]

\[0 < \theta < 1\]

(5)
Similarly, central approx. of $(U_{xx})_i^n$

\[ (U_{xx})_i^n = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} - \frac{h^2}{12} (U_{xxxx})_{i+\frac{1}{2}}^n \] (6)

Subst. (5) - (6) into (1)

\[ \frac{U_{i+1}^n - U_i^n}{k} - \frac{\sigma}{2} (U_{tt})_{i}^{n+1} = \frac{1}{h^2} \left[ \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} - \frac{h^2}{12} (U_{xxxx})_{i+\frac{1}{2}}^n \right] \]

Neglecting discretization errors, it leads to

\[ \frac{U_i^{n+1} - U_i^n}{k} = \frac{\sigma}{h^2} (U_{i+1}^n - 2U_i^n + U_{i-1}^n) \] (6.1)

Explicit method

\[ U_i^{n+1} = U_i^n + \frac{\sigma k}{h^2} (U_{i+1}^n - 2U_i^n + U_{i-1}^n), \quad i = 1, 2, \ldots, m \] (7)

FT-CS finite difference method for Heat Condu. (1-D).

STENCIL

Explain "marching in time" process.
by calling $r = \frac{\sigma k}{h^2}$ (7) Can be written as

$$U_{i}^{n+1} = r U_{i-1}^{n} + (1-2r) U_{i}^{n} + r U_{i+1}^{n}$$

(7.1)

$\quad i=1,2,...,m$

**Interior points**

**Explicit scheme**

**BCs:** $U_{0}^{n} = G_{0}(tn)$, $U_{m+1}^{n} = G_{1}(tn)$, $n = 1,2,...,N_{max}$

**IC:** $U_{i}^{0} = \eta (x_{i})$, $i = 1,2,...,m$

**Basic Experiment:** Example 2.3.1

$$U_{t} = \sigma U_{xx}, \quad 0 < x < 1, \quad t > 0$$

$$U(0,t) = 0, \quad U(1,t) = 0, \quad t > 0$$

$$U(x,0) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \leq x \leq 1 \end{cases}$$

**Exact Soln:**

$$U(x,t) = \sum_{k=1}^{\infty} \frac{8 \sin \left(\frac{k\pi x}{2}\right)}{(k\pi)^2} e^{-k^2 \pi^2 t} \sin (k\pi x).$$
Run the following experiments:

**Book Exp. 1:**

\[
\begin{align*}
\Delta t &= 0.001 \\
\Delta x &= 0.1 \\
\sigma &= 1
\end{align*} \Rightarrow r = \frac{\sigma \Delta t}{\Delta x^2} = 0.1
\]

**Book Exp. 2:**

\[
\begin{align*}
\Delta t &= 0.01 \\
\Delta x &= 0.1 \\
\sigma &= 1
\end{align*} \Rightarrow r = \frac{\sigma \Delta t}{\Delta x^2} = 1
\]

**Other Experiments:**

**Experiment 3:**

\[
\begin{align*}
\Delta t &= 0.0008 \\
\Delta x &= 0.05 \\
\sigma &= 1
\end{align*} \Rightarrow r = 0.32
\]

**Experiment 4:**

\[
\begin{align*}
\Delta t &= 0.001 \\
\Delta x &= 0.05 \\
\sigma &= 0.3 \\
t_{\text{max}} &= 0.3
\end{align*} \Rightarrow r = 0.4
\]

**Experiment 5:**

\[
\begin{align*}
\Delta t &= 0.0012 \\
\Delta x &= 0.05 \\
\sigma &= 1 \\
t_{\text{max}} &= 0.3
\end{align*} \Rightarrow r = 0.48
\]

**Exp. 6:**

\[
\begin{align*}
\Delta t &= 0.00125 \\
\Delta x &= 0.05 \\
\sigma &= 1 \\
t_{\text{max}} &= 0.3
\end{align*} \Rightarrow r = 0.5
\]

**Exp. 7:**

\[
\begin{align*}
\Delta t &= 0.00129 \\
\Delta x &= 0.05 \\
\sigma &= 1 \\
\end{align*} \Downarrow \Rightarrow r = 0.516
Local truncation error and consistency

Let's define

Continuous differential operator: \[ P_u = u_t - \sigma u_{xx} \] (8)

Discrete finite difference operator:

\[ P^\Delta u_i^n = \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta t} - \sigma \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{\Delta x^2} \] (9)

By substituting a sufficiently smooth function \( u(x, \varepsilon) \) into (9)

\[ P^\Delta u_i^n = \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta t} - \sigma \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{\Delta x^2} \] (10)
Now,
\[
\frac{V_i^{n+1} - V_i^n}{\Delta t} = \left(\frac{V_{ee}}{\Delta t}\right)_i + \frac{K}{2} \left(\frac{V_{ee}}{\Delta t}\right)_i^{n+1} \tag{11}
\]
\[
\frac{V_{ix}^n - 2V_i^n + V_{i-1}^n}{\Delta x^2} = \left(V_{xx}\right)_i^n + \frac{\Delta t}{12} \left(V_{xxxx}\right)_{i+\frac{1}{2}}^n \tag{12}
\]

Therefore, substituting (11) and (12) into (10)

\[
P_{V_i} V_i^n = \left(\frac{V_{ee}}{\Delta t}\right)_i^n + \frac{K}{2} \left(\frac{V_{ee}}{\Delta t}\right)_i^{n+1} - \sigma \left(V_{xx}\right)_i^n - \sigma \frac{\Delta t}{12} \left(V_{xxxx}\right)_{i+\frac{1}{2}}^n
\]

or

\[
P_{V_i} V_i^n = P_v(x_i; t^n) + \left[ \frac{K}{2} \left(\frac{V_{ee}}{\Delta t}\right)_i^{n+1} - \sigma \frac{\Delta t}{12} \left(V_{xxxx}\right)_{i+\frac{1}{2}}^n \right]
\]

\[
\Rightarrow P_v(x_i; t^n) - P_{V_i} V_i^n = -\frac{K}{2} \left(\frac{V_{ee}}{\Delta t}\right)_i^{n+1} + \sigma \frac{\Delta t}{12} \left(V_{xxxx}\right)_{i+\frac{1}{2}}^n
\]

**Definition:** The local discretization error of the FT-CS difference approx. for the heat equation differential operator is

\[
\tau_i^h = P_v(x_i; t^n) - P_{V_i} V_i^n = -\frac{K}{2} \left(\frac{V_{ee}}{\Delta t}\right)_i^{n+1} + \sigma \frac{\Delta t}{12} \left(V_{xxxx}\right)_{i+\frac{1}{2}}^n
\]
In general, if $P$ is a continuous differential operator and $P_\Delta V_i^n$ is a discrete finite difference operator.

**Definition.** The local discretization error of the finite difference $P_\Delta V_i^n$ approximation for the continuous operator $P$ is

$$T_i^n = P_\Delta V_i^n(x_i, t_n) - P V_i^n$$  \hspace{1cm} (5.0)

**Differential Equation defined by differential operator "L"**

$$P_u(x,t) = 0$$ \hspace{1cm} (5.1)

**Finite difference scheme defined by discrete finite diff. operator "$P_\Delta$"**

$$P_\Delta U_i^n = 0$$ \hspace{1cm} (5.2)

**Definition.** (Consistency)

A finite difference scheme (5.2) is consistent with a PDE (5.1), if the local discretization error tends to zero as $\Delta x \to 0$ and $\Delta t \to 0$. 
Convergence

Definition:

A finite difference approximation \( \tilde{U}^n = \begin{bmatrix} U^n_0 \\ U^n_1 \\ \vdots \\ U^n_J \end{bmatrix} \) converges to the solution of a partial differential equation

\[
\tilde{U}^n = \begin{bmatrix} U^n_0 \\ U^n_1 \\ \vdots \\ U^n_J \end{bmatrix}
\]

subject to initial or boundary conditions,

\[ U^n_j = U(x_j, t^n) \]

on a time \( 0 \leq t \leq T \) in a particular vector norm if

\[
\| \tilde{U}^n - \tilde{U}^n \| \to 0, \quad n \to \infty, \quad \Delta x \to 0, \quad \Delta t \to 0, \quad n \Delta t \leq T.
\]

Remark: Convergence implies that the discrete and continuous solutions approach each other for \( t \in (0, T) \) in a particular vector norm as the mesh spacing decreases.
Convergence of FT-CS scheme for heat equation in $\Omega \times (0, \infty)$. 

\[
\begin{cases}
U_t = \sigma U_{xx}, & t > 0, \quad 0 < x < 1, \\
U(x,0) = \phi(x), \\
U(0,t) = g(t), \quad U(1,t) = h(t).
\end{cases}
\]

**FT-CS finite difference approximation.** 

\[
\frac{U_j^{n+1} - U_j^n}{\Delta t} - \sigma \left[ \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} \right] = \frac{\Delta t}{2} \left( U_{tt} \right)_j^n - \frac{\sigma \Delta x^2}{12} \left( U_{xxxx} \right)_j^n
\]

\[
0 < \sigma < 1, \quad 0 < \Delta x < 1
\]

Local truncation error is defined as 

\[
\tau_j^n = -\frac{\Delta t}{2} (U_{tt})_j^n + \sigma \frac{\Delta x^2}{12} (U_{xxxx})_j^n
\]  

(1.2)

Solving for $U_j^{n+1}$ in (1.1). 

\[
U_j^{n+1} = r U_j^n + (1-2r) U_{j-1}^n + r U_{j+1}^n - \Delta t \tau_j^n
\]  

(1.3)

Also neglecting the L.T.E., we obtain the numerical scheme. 

\[
\begin{cases}
U_j^{n+1} = r U_j^n + (1-2r) U_{j-1}^n + r U_{j+1}^n, \\
\end{cases}
\]  

(1.4)

\[
\begin{align*}
\tau_j = \tau_j^n, & \quad \tau_0^n = g(t_n), \quad \tau_{J+1}^n = h(t_n) = h_n, \\
U_j^0 = \phi(x_j) = \phi_j.
\end{align*}
\]
Subtracting (1.4) from (1.3), and calling

\[ e_j^n = y_j^n - y_j^{n+1}, \]

we obtain

\[ e_j^{n+1} = r \frac{e_j^n}{e_{j+1} - e_{j-1}} e_j^n + r e_j^{n+1} - \Delta t \dot{y}_j^n \]

Applying triangular inequality

\[ |e_j^{n+1}| \leq r |e_{j+1}^n| + |r - 1| |e_j^n| + r |e_{j+1}^n| + \Delta t |\dot{y}_j^n| \tag{\star} \]

Theorem: FT-CS for heat conduction eqn. converges in max norm when \( r < \frac{1}{2} \).

In other words, we want to prove

that \( \|\ddot{e}\|_\infty \to 0 \), when \( \Delta x \to 0, \Delta t \to 0 \) as \( n \to \infty \).

where

\[ \ddot{e}^n = \begin{bmatrix} e_1^n \\ e_2^n \\ \vdots \\ e_{n+1}^n \end{bmatrix} \]

Remark: Notice that \( e_0^n = 0 \) and \( e_1^n = 0 \), for all \( n \).
Assuming there is not rounding errors at the computation in the boundaries.
If \( \|e^n\|_\infty = \max_j |e^n_j| \), \( \|\tilde{r}^n\|_\infty = \max_j |\tilde{r}^n_j| \)

then,
\[
|e^{n+1}_j| \leq (r + |1-2r| + r) \|\tilde{e}^n\|_\infty + \Delta t \|\tilde{r}^n\|_\infty \quad j = 1, 2, \ldots, J-1
\]

If \( r \leq \frac{1}{2} \Rightarrow 1-2r \geq 0 \) and \( |1-2r| = 1-2r \)
\[
|e^{n+1}_j| \leq \|\tilde{e}^n\|_\infty + \Delta t \|\tilde{r}^n\|_\infty \quad j = 1, 2, \ldots, J-1
\]

\[ \Rightarrow \frac{\|\tilde{e}^{n+1}\|}{\theta} \leq \frac{\|\tilde{e}^n\|}{\theta} + \Delta t \frac{\|\tilde{r}^n\|}{\theta} \qquad (\star \star) \]

Iterating
\[
\|\tilde{e}^{n+1}\|_\infty \leq \|\tilde{e}^0\|_\infty + \Delta t (\|\tilde{r}^0\|_\infty + \|\tilde{r}^{n-1}\|_\infty + \ldots + \|\tilde{r}^{n-(n-1)}\|_\infty) \leq \|\tilde{e}^0\|_\infty + \Delta t \sum_{k=0}^{n-1} \|\tilde{r}^k\|_\infty \Delta t
\]

where \( r \overset{\text{def}}{=} \max_{0 \leq k \leq n} \|\tilde{r}^k\|_\infty \)

Since \( \|\tilde{e}^0\|_\infty \to 0 \) and \( n \Delta t \leq T \Rightarrow \|\tilde{e}^n\| \leq r \Delta t \)

\[ \Rightarrow \|\tilde{e}^n\|_\infty \leq Tr \]

\[ r \overset{\text{def}}{=} \frac{\sigma \Delta t}{\Delta x^2} \leq \frac{1}{2} \Rightarrow \Delta t \leq \frac{1}{2 \sigma} (\Delta x)^2 \]

Severe restriction in time step, since \((\Delta x)^2\) may be very small.
Now
\[ T \leq \frac{\Delta t}{\sigma} K + \sigma \frac{\Delta x^2}{12} M \]

using definition of truncation error (1.2)
\[ K = \max_{0 \leq t \leq T} \left\| u_{t+1} \right\|, \quad M = \max_{0 \leq t \leq T} \left\| u_{xxx} \right\|/\]

\[ \Rightarrow \left\| \hat{e}^n \right\|_\infty \leq T \left( \frac{\Delta t}{\sigma} K + \sigma \frac{\Delta x^2}{12} M \right) \to 0 \]

\[ \Delta t \to 0, \quad \Delta x \to 0, \quad n \to \infty \]

and \( n \Delta t \leq T \)

---

Remark: The inequality (*) is different for \( j = 1 \) and \( j = J-1 \).

In fact,
\[ r \leq \frac{1}{2} . \]

For \( j = 1 \):
\[ |e_j^n| \leq r |e_0^n| + (1-2r) |e_1^n| + r |e_2^n| + \Delta t |\hat{T}_1^n| \]

For \( j = J-1 \):
\[ |e_{J-1}^n| \leq r |e_{J-2}^n| + (1-2r) |e_{J-1}^n| + r |\hat{e}_J^n| + \Delta t |\hat{T}_J^n| \]

In both cases:
\[ j = 1 \quad \text{or} \quad \sqrt{e_j^n} \leq (1-r) \sqrt{\tilde{T}_1^n} + \Delta t \sqrt{\hat{T}_1^n} \leq \sqrt{e_0^n} + \Delta t \sqrt{\hat{T}_0^n} \]

Same as (**) So, we proceed identically from here.
The condition \( R = 1/2 \) impose limitations on the choice of \( \Delta t \).

How can we define numerical schemes for our IBVP with less limitation on the choice of \( \Delta t \)?

Idea: Domain of dependence for num. sch. (2)

![Diagram showing the domain of dependence and the relationship between \( \tan \theta \) and \( \Delta x / \Delta t \).]

Obviously, boundary values at points Q and R at level \( n \) don't enter into the computation of P at level \( n \).
From PDE theory, we know that solution at point $P$ certainly depends on boundary data at $Q$ and $R$.

From the previous graph, we conclude that the angle $\Theta$ should be $\pi/2$ (or close to it) for $Q$ and $R$ to enter into the computation at $P$.

In previous work, we performed two experiments depending on $r$: values, for FT-CS scheme.

a) $r = 10^{-1}$, was stable and converges.

b) $r = 1$, Num. Sch. unstable.

In (a), $\Theta = \tan^{-1} \left( \frac{\Delta x}{\Delta t} \right) = \tan^{-1} \left( \frac{\sigma}{r \Delta x} \right)$, $r = 10^{-1}$.

\[ \Theta = \tan^{-1} \left( \frac{1}{10^{-1} \cdot 10^{-3}} \right) = \tan^{-1} (100) \approx 1.56 \approx \pi/2 \]

In (b), $r = 1$.

\[ \Theta = \tan^{-1} \left( \frac{1}{1 \times 10^{-1}} \right) = \tan^{-1} (10) \approx 1.47 < \pi/2 \]
The previous analysis motivates the construction of implicit schemes. For implicit schemes, the solution at \( P \) will involve all the other unknowns at the same time level, and it will also include the boundary conditions at \( Q \) and \( R \).

Example: BT-C5 at the point \((x_j, t_{n+1})\)

\[
(U_t)_{j}^{n+1} = \sigma (U_{xx})_{j}^{n+1}
\]

Approx. by

\[
\frac{U_j^{n+1} - U_j^n}{\Delta t} = \sigma \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{\Delta x^2}
\]

\( j = 1, 2, \ldots, J-1 \)

Also called backward-Euler method.

It can be written as

\[
\begin{bmatrix}
-rU_{j-1}^{n+1} + (1+2r)U_j^{n+1} - rU_{j+1}^{n+1} = U_j^n
\end{bmatrix}
\]

\( j = 1, 2, \ldots, J-1 \)

For our IBVP (1), we also know

\[
U_0^{n+1} = G(t_{n+1}) = \bar{g}_{n+1}, \quad U_J^{n+1} = \bar{h}_{n+1}
\]
Computational stencil:

Obviously, for a given \( J \) equ. (4.1) is not enough. A system of equations needs to be solved at every time level \( n+1 \).

In particular, if \( J = 4 \)

We choose a system of 3 equs. to be solved simultaneously. In fact,

\[
\begin{align*}
  j &= 1, & -r U_1^{n+1} + (1+2r) U_1^n - r U_2^{n+1} &= V_1^n \\
  j &= 2, & -r U_1^n + (1+2r) U_2^{n+1} - r U_3^n &= U_2^n \\
  j &= 3, & -r U_2^{n+1} + (1+2r) U_3^{n+1} - r U_4^{n+1} &= U_3^n 
\end{align*}
\]
With BCs:
\[ U_0^{n+1} = g^{n+1}, \quad U_4^{n+1} = h^{n+1} \]

The above system can be written in matrix form as

\[
\begin{pmatrix}
1 + 2r & -r & 0 \\
-r & 1 + 2r & -r \\
0 & -r & 1 + 2r
\end{pmatrix}
\begin{pmatrix}
U_1^{n+1} \\
U_2^{n+1} \\
U_3^{n+1}
\end{pmatrix}
= \begin{pmatrix}
U_1^n + rg^{n+1} \\
U_2^n \\
U_3^n + rh^{n+1}
\end{pmatrix}
\]

For a larger partition: \( j = 1, \ldots, J-1 \)
\( n = 1, \ldots, N \)

A linear system for the unknowns: \( U_1^n, U_2^n, \ldots, U_J^n \) at each time level \( t_n \) needs to be solved.

\[ A \tilde{U}^{n+1} = \tilde{F}^n \]

Where
\[ A = \begin{bmatrix}
1 + 2r & -r & 0 & 0 & \cdots & 0 \\
-r & 1 + 2r & -r & 0 & \cdots & 0 \\
0 & -r & 1 + 2r & -r & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & -r & -r & 1 + 2r & -r \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}_{(J-1) \times (J-1)} \]

\[ \dot{U}^{n+1} = \begin{bmatrix}
U_1^{n+1} \\
U_2^{n+1} \\
\vdots \\
U_m^{n+1} \\
\end{bmatrix}_{(J-1) \times 1}, \quad \dot{F}^n = \begin{bmatrix}
U_1^n + r g^{n+1} \\
U_2^n \\
U_3^n \\
\vdots \\
U_m^n + r f^{n+1} \\
\end{bmatrix}_{(J-1) \times 1} \]

At every time step, this system is solved.
Another Implicit Scheme:

Crank-Nicholson method:

\[-\frac{r}{2} U_i^{n+1} + (1+r) U_i^{n+1} - \frac{r}{2} U_i^{n+1} = \frac{r}{2} U_i^n + (1-r) U_i^n + \frac{r}{2} U_i^{n+1} \]

\[i = 1, 2, \ldots, m \]

Derivation:

Approximate \( U_t \) and \( U_{xx} \) at the point \((x_i, t_{n+1/2})\)

Using centered differences in both time and space

With time step size \( \Delta t \) and space step size \( \Delta x \):

\[ (U_t)^{n+1/2}_{i} = \sigma (U_{xx})_{i}^{n+1/2} \]

\[ \text{CT-CS} \]

\[ \left( \frac{\text{BE}}{\Delta x} \right) \]

\[ \frac{U_i^{n+1} - U_i^n}{\Delta t} = \sigma \frac{U_{i+1}^{n+1/2} - 2U_i^{n+1/2} + U_{i-1}^{n+1/2}}{\Delta x^2} \]
Using average in time

\[ \frac{U_i^{n+1} - U_i^n}{\kappa} = \sigma \left( \frac{U_{i+1}^{n+1} + U_i^{n+1}}{2} - 2 \frac{U_i^{n+1} + U_i^n}{2} + \frac{U_{i-1}^{n+1} + U_i^n}{2} \right) \]

\[ \Rightarrow \]

\[ U_i^{n+1} - U_i^n = \frac{\sigma \kappa}{2h^2} \left[ U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_i^{n+1} - 2U_i^n + U_i^n \right] \]

\[ r = \frac{\sigma \kappa}{h} \]

\[ \frac{-r}{\Delta x} U_{i-1} + (1+r) U_i^{n+1} - \frac{r}{\Delta x} U_{i+1}^{n+1} = \]

\[ = \frac{r}{\Delta x} U_i^n + (1-r) U_i^n + \frac{r}{\Delta x} U_{i+1}^n \]

\[ i = 1, \ldots, m \]

In matrix form:

\[ A \hat{U}^{n+1} = B \hat{U}^n + (C \hat{D}) \]

(11.1)

\[ \Rightarrow \text{due to BCs at } i=0, \text{ and } i=m+1. \]

Remark:

Matrix Equation (11.1) needs to be solved at each time level "n".
where

\[
A = \begin{bmatrix}
1 + r & -r/2 & 0 & 0 & \cdots & 0 \\
-r/2 & 1 + r & -r/2 & 0 & \cdots & 0 \\
0 & -r/2 & 1 + r & -r/2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -r/2 & 1 + r & -r/2 \\
0 & 0 & \cdots & 0 & -r/2 & 1 + r \\
\end{bmatrix}_{m \times m}
\]

\[
B = \begin{bmatrix}
1 - r & r/2 & 0 & 0 & \cdots & 0 \\
r/2 & 1 - r & r/2 & 0 & \cdots & 0 \\
0 & r/2 & 1 - r & r/2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & r/2 & 1 - r & r/2 \\
r/2 & 0 & \cdots & 0 & r/2 & 1 - r \\
\end{bmatrix}_{m \times m}
\]

\[
\bar{C}^n = \begin{bmatrix}
\frac{r}{2} g^{n+1} + \frac{r}{2} g^n \\
0 \\
\vdots \\
0 \\
\frac{r}{2} h^{n+1} + \frac{r}{2} h^n
\end{bmatrix}
\]

\[
\bar{U}^n = \begin{bmatrix}
U_1^n \\
U_2^n \\
\vdots \\
U_m^n
\end{bmatrix}
\]
Notice that the FT-CS difference scheme for heat conduction can be expressed in matrix form:

\[
\begin{bmatrix}
U_1^{n+1} \\
U_2^{n+1} \\
\vdots \\
U_{j-1}^{n+1}
\end{bmatrix}
= \begin{bmatrix}
1-2r & r & 0 & \cdots & 0 \\
r & 1-2r & r & \cdots & 0 \\
0 & r & 1-2r & r & \cdots \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & r & 1-2r
\end{bmatrix}
\begin{bmatrix}
U_1^n \\
U_2^n \\
\vdots \\
U_{j-1}^n
\end{bmatrix}
\]

or \( \hat{\mathbf{U}}^{n+1} = \mathbf{L}_{\Delta} \hat{\mathbf{U}}^n \).

**Stability**

**Definition:** Consider two different initial value problems for the same finite difference scheme, i.e.,

\[
\hat{\mathbf{U}}^{n+1} = \mathbf{L}_{\Delta} \hat{\mathbf{U}}^n, \quad \hat{\mathbf{U}}^0 = \phi
\]

\[
\hat{\mathbf{V}}^{n+1} = \mathbf{L}_{\Delta} \hat{\mathbf{V}}^n, \quad \hat{\mathbf{V}}^0 = \psi
\]

This finite difference scheme is stable if there exists a positive constant \( C \), independent of the mesh spacing and initial data, such that

\[ \| \hat{\mathbf{U}}^n - \hat{\mathbf{V}}^n \| \leq C \| \hat{\mathbf{U}}^0 - \hat{\mathbf{V}}^0 \|, \quad n \to \infty, \ \Delta x \to 0, \ \Delta t \to 0, \ \text{not} \leq T. \]
If $L_\Delta$ is linear, the definition of stability can be written as

**Definition:** A finite difference scheme

$$U^{n+1}_\Delta = L_\Delta U^n_\Delta,$$

for a homogeneous IVP $U^n_\Delta$, is stable if there exists a positive constant $C$, independent of the mesh spacing and initial data such that

$$\|U^n_\Delta\| \leq C \|U^0_\Delta\|, \ n \to \infty, \ \delta x \to 0, \ \delta t \to 0, \ n \delta t \leq T.$$

**Remark:** When $L_\Delta$ is linear the two definitions are equivalent.

**Maximum principle**

Finite difference schemes as the FT-CS and Crank-Nicholson for the heat equation are called one-level finite difference schemes because they only involve solutions at time levels $n$ and $n+1$. 
Theorem.

A sufficient condition for stability of the one-level finite difference scheme

\[ U_j^{n+1} = \sum_{|s| \leq S} C_s U_{j+s}^n \]  

in the \( \| \cdot \|_{\infty} \) is that all coefficients \( C_s \) \( (|s| \leq S) \) be positive and add to unity.

Proposition

The FT-CS finite difference scheme applied to a homogeneous IVP is stable if \( r \leq \frac{1}{2} \).

Proof:

\[ U_j^{n+1} = r U_{j-1}^n + (1 - 2r) U_j^n + r U_{j+1}^n \]

Then

\[ \sum_{|s| \leq S} C_s = r + (1 - 2r) + r = 1 \]

Since \( r \leq \frac{1}{2} \)

\[ r = \sigma \frac{\Delta t}{\Delta x^2} > 0. \]
Proof of Theorem

Using triangular inequality in (*)

\[ |U_j^{n+1}| \leq \sum_{k=1}^{s} |C_k| |U_{j+k}^n|, \quad j = 1, 2, \ldots, s-1 \]

introducing \( || . ||_\infty \)

\[ ||U^{n-1}|| = \left( \sum_{k=1}^{s} C_k \right) ||U^n|| = ||U^n|| \]

\[ \Rightarrow ||U^n|| \leq ||U^{n-1}|| \leq \ldots \leq ||U^0||, \quad n \to \infty, \Delta x \to 0, \Delta t \to 0, \quad n \Delta t \leq T. \]

what happens if \( \sum_{k=1}^{s} C_k = C > 1 \) ?

Order of Numerical Scheme

Definition: A consistent finite-difference scheme approximating a partial differential equation is of order \( p \) in time and order \( q \) in space if

\[ T_j^n = O(\Delta t^p) + O(\Delta x^q). \]