2.2 Simple Differences Schemes for a Kinematic Wave Equation.

\[
\begin{align*}
    \mu x + A(u) \mu x &= 0, \quad -\infty < x < \infty, \quad t > 0 \quad (1) \\
    \mu(x,0) &= \phi(x), \quad -\infty < x < \infty. \quad (2)
\end{align*}
\]

For computational purposes, we need a finite domain. This can be accomplished by asking

\[\phi(x) \equiv 0, \quad |x| > \overline{X}\]

or be periodic

\[\phi(x+\overline{X}) = \phi(x), \quad \overline{X} > 0.\]

We will use forward finite differences to approximate the derivatives in (1).

Neglecting the local discretization errors, we obtain

\[
\frac{U^n_{i+1} - U^n_i}{\Delta t} + A(U^n_i) \frac{U^n_{i+1} - U^n_i}{\Delta x} = 0.
\]
\[ y_j^{n+1} = y_j^n - \frac{\Delta t}{\Delta x} a(y_j^n) [ y_j^n - y_j^{n-1} ] = 0 \]

or

\[
\frac{y_j^{n+1}}{y_j^n} = (1 + \sigma_j^n) y_j^n - \sigma_j^n y_j^{n-1}
\]

\[ \text{FORMAL-TIME AND FORWARD-SPACE} \]

The number \( \sigma_j^n = a(y_j^n) \frac{\Delta t}{\Delta x} \) is called the Courant number.

Formula (3.1) involves 3 points.

\[ y_j^{n+1} = (1 - \sigma_j^n) y_j^n + \sigma_j^n y_j^{n-1} \]

\[ \text{FORMAL-TIME BACKWARD SPACE} \]

\[ \text{FORMAL-TIME BACKWARD SPACE} \]
Forward Time and Centered Space

\[ v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n) \]

Verify it! (3.1)

Example 3.2.1

\[
\begin{cases}
    u_t + a u_x = 0 \\
    u(x,0) = \phi(x) = \begin{cases} 
        x, & x < 0 \\
        0, & x > 0
    \end{cases}
\end{cases}
\]

\(-\infty < x < \infty, t > 0\) (3)

It has exact solution: \(u(x,t) = \phi(x-t)\).

Obtain a numerical solution for this IVP using

\(\Delta x = \frac{1}{10}, \ \Delta t = \frac{1}{20}\)

Show MATLAB Code

and Graphs of the numerical solution.
Domain of Dependence.

(1) \[ u_t + au_x = 0, \quad u(x,0) = \phi(x) \quad (1) \]

\[ a > 0 \]

Domain of dep. of (1) at \((x,t)\) is the point \(x_0 = x - at\).

Since this point determine the soln. at \((x,t)\).

(II) FT-BS: \[ u_{j}^{n+1} = (1-d_j)u_{j}^{n} + d_j u_{j-1}^{n} \quad (2) \]

Domain of dep. of num. scheme (2) \[ \Delta = \frac{c \Delta t}{\Delta x} \]

at \((x_j, t_n)\) is \([ (j-n)\Delta x, j\Delta x] \)

For instance,

\[ n = 3 \]

\[ (x_4, t_3) \]

\[ n = 0 \]

\[ g = 1 \]

\[ g = 4 \]

Domain of dep. at \((x_4, t_3)\) of num. scheme (2) is \([ \Delta x, 4\Delta x] \)
CFL Theorem.

A necessary condition for the convergence of the solution of a finite difference approximation to the solution of (1) for arbitrary initial data is that

\[ \text{Domain of } \Delta t \text{ of } \text{finite difference approx.} \subset \text{Domain of } \Delta t \text{ of } \text{IVP (1)} \]

\[ \frac{\Delta t}{\Delta x} \leq 1 \]

Necessary condition for convergence:

Courant number \( \frac{1}{a} \geq \frac{\Delta t}{\Delta x} \Rightarrow a \leq \frac{\Delta x}{\Delta t} \)

or \( \left( \frac{\Delta t}{\Delta x} \right) \leq 1 \)

phys. wave speed

Num wave speed

Domain of dep. of finite difference approx.

Contains domain of dependence of IVP (1).

Logical statement and implications:

Convergence \( \Rightarrow \) CFL cond.

7 CFL cond \( \Rightarrow \) 7 convergence.
\[ \frac{FT-FS}{\Delta t} : \quad \frac{U_i^{n+1}}{\Delta t} = (1 + \alpha_j^n) \frac{U_j^n - U_{j+1}^n}{\Delta x} \quad \text{(3)} \]

\begin{align*}
\frac{\Delta x}{\Delta t} & = \frac{x_5 - x_3}{\Delta t} = x_0 \\
J & = 5 \quad j = 6 \quad j = 9
\end{align*}

Domain of dep. Num. scheme
\[ [5\Delta x, 9\Delta x] \]

\( x_0 \in [5\Delta x, 9\Delta x] \) \quad CFL is not satisfied.

Therefore, Soln. of (3) does not converge to the Soln. of (1) for arbitrary I.c.

CFL Thm can be proved by assuming no CFL and that
\[ \phi(x) = \begin{cases} 
0, & \text{in dom. of dep. N.S. for } (x,t) \\
1, & \text{in dom. of dep. IVP. for } (x,t) 
\end{cases} \]

Then, Soln. of N.S. is zero at \( (x,t) \), while Soln of IVP is nonzero at \( (x,t) \).
If we consider the IVP:

\[ u_t + au_x = 0, \quad u(x,0) = \varphi(x) \]

Then

Also, FT-FS scheme (3)

\[
\text{Slope} = \frac{t_{n+1} - t_n}{x_{j+1} - x_j} = \frac{\Delta t}{-\Delta x}
\]

So CFL condition is \( x_0 \in [j\Delta x, (j+1)\Delta x] \)

which is equivalent to

\[
\frac{1}{a} \leq -\frac{\Delta t}{\Delta x} \iff a \geq \frac{\Delta x}{\Delta t}
\]

\[
\left\lfloor \frac{\Delta x}{\Delta t} \right\rfloor \geq 1
\]

Cannot #