Iterative Methods Applied to Poisson's Eq.

Continuous:
\[ \begin{cases} \nabla^2 U = f(x,y) \\ U(x_0) = g(x_0), \ x_0 \in \Box \end{cases} \] (1)

Discrete 5-point FDM:
\[ \nabla^2 U = f_{ij} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{\Delta y^2} = f_{ij} \] (2)

or
\[ -2 \left( \frac{\Delta y^2 + \Delta x^2}{(\Delta x)^2 (\Delta y)^2} \right) U_{ij} + \frac{1}{(\Delta x)^2} (U_{i+1,j} + U_{i-1,j}) + \frac{1}{(\Delta y)^2} (U_{i,j+1} + U_{i,j-1}) = f_{ij} \] (3)

Then,
\[ U_{ij} = \frac{(\Delta y)^2}{2(\Delta x^2 + \Delta y^2)} (U_{i+1,j} + U_{i-1,j}) + \frac{(\Delta x)^2}{2(\Delta x^2 + \Delta y^2)} (U_{i,j+1} + U_{i,j-1}) - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} f_{ij} \]

or
\[ U_{ij} = \Theta_x (U_{i+1,j} + U_{i-1,j}) + \Theta_y (U_{i,j+1} + U_{i,j-1}) - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} f_{ij} \] (4)

If \[ \Delta x = \Delta y \] \[ \Theta_x = \frac{1}{4}, \Theta_y = \frac{1}{4} \]

\[ \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} = \frac{(\Delta x)^2}{4} \]

or
\[ U_{ij} = \frac{1}{4} (U_{i+1,j} + U_{i-1,j}) + \frac{1}{4} (U_{i,j+1} + U_{i,j-1}) - \frac{(\Delta x)^2}{4} f_{ij} \] (5)
Two methods:

i) Direct: Solve system of equations obtained from (4) or (5)

\[ A \hat{U} = \hat{F} \]

Where \( A \) is given as discussed before.

It can be shown \( A \) is nonsingular

ii) Iterative:

A) Jacobi:

\[ U_{ij}^{(k+1)} = \frac{1}{4} \left[ U_{i+1,j}^{(k)} + U_{i-1,j}^{(k)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k)} \right] - \frac{h^2}{4} f_{ij} \]

It can be shown that it converges for any initial guess: \( \hat{U}^{(0)} \).

b) Gauss-Seidel:

\[ U_{ij}^{(k+1)} = \frac{1}{4} \left[ U_{i+1,j}^{(k)} + U_{i-1,j}^{(k)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k)} \right] - \frac{h^2}{4} f_{ij} \]

Go to general discussion with elementary matrices (3x3).
Complexity:

- A is never stored. Storage is optimal
- Only $m^2$ in Gauss-Seidel, for solution vector $2m^2$ in Jacobi
- Each iteration requires $O(m^2)$ work
- \# iterations = $O(m^2 \log m)$

Then, work/iter = $O(m^4 \log m)$

Worst than Gauss elim. $O(m^3)$

with banded solver $O((m^2)^3)$

$O(m^6)$ not banded.

Other with same $O(m^2)$ work/iter converge faster.

If converging is indep. of $h$
then total work = $O(m^2)$ multigrid methods.

for many elliptic problems.