Acoustic Wave equation

Derivation:

Unknown functions involved:

\( \tilde{u}(\tilde{x},t) \): Velocity.
\( \rho(\tilde{x},t) \): Density.
\( P(\tilde{x},t) \): Pressure, \( \mathcal{E}(\tilde{x},t) \): Entropy.
\( f(\tilde{x},t) \): External force.

The fundamental hypothesis in linear acoustic is that there exists small perturbations about an equilibrium state. It's assumed an equilibrium state where

\[
\rho = \rho_0(\tilde{x}, \gamma, \varepsilon) = \rho_0(\tilde{x}), \text{ independent of } t.
\]

\[
P = P_0(\text{constant}).
\]

\[
\tilde{u} = \tilde{0}, \text{ and } \tilde{f} = \tilde{0}.
\]

We define asymptotic expansions about the equilibrium.

\[
\tilde{u}(\tilde{x},t) = \tilde{0} + \varepsilon \tilde{u}'(\tilde{x},t) + O(\varepsilon^2).
\]

\[
P(\tilde{x},t) = P_0 + \varepsilon P'(\tilde{x},t) + O(\varepsilon^2).
\]

\[
P(\tilde{x},t) = P_0 + \varepsilon P'(\tilde{x},t) + O(\varepsilon^2).
\]

\[
\tilde{f}(\tilde{x},t) = \tilde{0} + \varepsilon \tilde{f}'(\tilde{x},t) + O(\varepsilon^2).
\]

(0)
Navier-Stokes equations govern the gas dynamic phenomena. They are conservation laws plus an equation of state.

Mass Conservation:

\[ \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0. \]  \hspace{1cm} (1)

Conservation of Momentum:

\[ \rho \left[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla P + \mathbf{f}. \]  \hspace{1cm} (2)

Conservation of Energy:

\[ \frac{DS}{Dt} = S_t + \mathbf{u} \cdot \nabla S = 0. \]  \hspace{1cm} (3)

Equation of State:

\[ S = S(P, \rho) = \frac{P}{\rho^\gamma}, \]  \hspace{1cm} (4)

where \( \gamma = \frac{C_p}{C_v} \) is the ratio of specific heats at pressure and constant volume, respectively.

Entropy is constant along particles path.
From (4), applying the chain rule
\[ S_t = S_{p} P_t + S_{p} P_{t} \quad \nabla S = S_{p} V P + S_{p} V P \]
Substitution into (3) leads to
\[ S_{p} P_t + S_{p} L_{t} + \ddot{u} \cdot (S_{p} V P + S_{p} V P) = \]
\[ = S_{p} [P_t + \ddot{u} \cdot V P] + S_{p} [P_t + \ddot{u} \cdot V P] = 0. \quad \text{(5)} \]
Defining \( C^2 (P, P) \equiv -\frac{S_{p}}{S_{p}} \quad \text{acoustic velocity} \)
(5) is transformed into
\[ \sqrt{P_t + \ddot{u} \cdot V P - C^2 [P_t + \ddot{u} \cdot V P]} = 0. \quad \text{(6)} \]
Equations (1), (2) and (6) form a nonlinear system of PDE that models gas dynamics phenomena.

By substituting the asymptotic expansions about the equilibrium (0) into (1), (2) and (6), we will be able to linearize our nonlinear PDE system.
Summarizing, we have obtained a linearized system of PDE for the unknowns: $\rho', p', \text{ and } \tilde{u}'$

\[
P_t' + \nabla \cdot (\rho_0 \tilde{u}') = 0, \tag{7}
\]

\[
\rho_0 \tilde{u}_t' = -\nabla p' + \tilde{f}', \tag{8}
\]

\[
P_t' - C_0^2 \left[ p_t' + \tilde{u}' \cdot \nabla p_0 \right] = 0 \tag{9}
\]

where

\[
C_0^2 = \frac{\frac{\rho_0}{\bar{\rho_0}}}{p_0}
\]

The next steps are to eliminate

1) $\tilde{u}'$ from (7) and (9). First, solve for $\tilde{u}'$ in (8) and substitute it into (7) and (9).

2) Eliminate $p'$ from the new equations (7) and (9) by combining them.

Therefore, a single scalar equation for the leading order pressure term, $P'(x,t)$, is obtained.
As a result of this combination, we obtain

**Acoustic wave equation for the pressure \( P \):**

\[
\frac{1}{C_0^2} \frac{\partial^2 P}{\partial t^2} - \rho_0 \nabla \cdot \left( \frac{1}{\rho_0} \nabla P' \right) = -\rho_0 \nabla \cdot \left( \frac{1}{\rho_0} \frac{\partial f'}{\partial t} \right) \tag{\text{x}}
\]

If \( \rho_0 = \text{Const.} \), \( \text{(x)} \) reduces to

\[
\nabla^2 P' - \frac{1}{C_0^2} \frac{\partial^2 P'}{\partial t^2} = \nabla \cdot \frac{\partial f'}{\partial t} \tag{\text{x} \times}
\]
If the external force is periodic in time,
\[ f'(x,t) = \tilde{F}(x) e^{-i\omega t} \]

It can be shown that after a transient state of erratic behavior due to the IC's, a time-harmonic state is reached. It means
\[ p'(x,t) = p(x) e^{-i\omega t} \]

Notice that
\[ p' = -i\omega p \quad e^{-i\omega t} \]
and
\[ p'' = -\omega^2 p \quad e^{-i\omega t} \]

Subst. in (10) leads to
\[ \frac{-\omega^2}{c_0^2} p e^{-i\omega t} - p_0 \nabla \left( \frac{1}{p_0} p \right) e^{-i\omega t} = -p_0 \nabla \left( \frac{1}{p_0} \tilde{F} \right) e^{-i\omega t} \]

or
\[ p_0 \nabla \left( \frac{1}{p_0} \nabla p \right) + \frac{\omega^2}{c_0^2} p = p_0 \nabla \left( \frac{1}{p_0} \tilde{F} \right) \quad (12) \]

It's called Reduced Wave equation.

If \( p_0 = \text{const.} \)
\[ \nabla^2 p + K_0^2 p = \nabla \cdot \tilde{F}, \quad \text{Helmholtz equation.} \quad (13) \]

where \( K_0 = \omega/c_0. \)