Numerical Solution of PDE

Winter Semester 2004

Midterm Project.

1) Acoustic Scattering from an Empty Cavity.

Consider the IBVP:

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi = c^2 \left[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right]$$  \hspace{1cm} (1)

**B.C. at obstacle**

$$\rho_{sc} (r, \theta, t) = -|\text{Pinc}(r, \theta, t)|$$ \hspace{1cm} (2)

**Rad. Cond.**

$$\frac{\partial \rho_{sc}}{\partial t} + c \frac{\partial \rho_{sc}}{\partial r} \to 0$$ \hspace{1cm} \text{r} \to \infty \hspace{1cm} (3)

**I.C.'s**

$$\begin{cases} \rho_{sc} (r, \theta, 0) = f(r, \theta) \\ \frac{\partial \rho_{sc}}{\partial t} (r, \theta, 0) = g(r, \theta) \end{cases}$$ \hspace{1cm} (4)

$$|\text{Pinc}(r, \theta, t)| = e^{i\pi r \cos(\theta - \alpha)} e^{-i\omega t}$$ \hspace{1cm} (6)
i) Show that \( \psi \) satisfies (1).

ii) Obtain a numerical approximation for \( \psi \) using a finite difference numerical scheme CT-CS. March iii time until a harmonic steady state is reached. (Limiting Amplitude Principle).

iii) You will need to provide your code with the following data:

- \( \alpha \): Incident angle. (\( \alpha = 0 \) is good to start).
- \( r_0 \): Obstacle radius. Take \( r_0 = 1 \).
- \( r_{\infty} \): Fictitious boundary at \( \infty \). Try with several values, e.g., \( r_{\infty} = 5, 9, 12 \).

<table>
<thead>
<tr>
<th>( \Delta r )</th>
<th>( \Delta \theta )</th>
<th>( \Delta t )</th>
<th>( \omega )</th>
<th>( k )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>radial step size</td>
<td>angular step size</td>
<td>time increment</td>
<td>time frequency</td>
<td>wave number</td>
<td>tolerance for stop criteria</td>
</tr>
<tr>
<td>0.1</td>
<td>( \pi/20 )</td>
<td>0.05</td>
<td>5</td>
<td>5</td>
<td>( 10^{-2} )</td>
</tr>
<tr>
<td>0.1</td>
<td>( \pi/40 )</td>
<td>0.05</td>
<td>5</td>
<td>5</td>
<td>( 10^{-2} )</td>
</tr>
<tr>
<td>0.1</td>
<td>( \pi/80 )</td>
<td>0.05</td>
<td>5</td>
<td>5</td>
<td>( 10^{-2} )</td>
</tr>
<tr>
<td>0.1</td>
<td>( \pi/180 )</td>
<td>0.01</td>
<td>5</td>
<td>5</td>
<td>( 10^{-3} )</td>
</tr>
<tr>
<td>0.1</td>
<td>( \pi/60 )</td>
<td>0.005</td>
<td>5</td>
<td>5</td>
<td>( 10^{-4} )</td>
</tr>
</tbody>
</table>

(Values may vary)

\( N_T \): Maximum number of time iterations.

You will need to adjust this for every experiment. Start with \( N_T = 10,000 \).
iv) Show that the data provided in the first 4 columns satisfy the stability condition (Constant number 2), but the one in column 5 doesn’t.

v) Assume IC’s: \( f(0) = 0 \), \( g(0) = 0 \). Why is this a good choice?

vi) You need to store three levels of time. Because your method is a multilevel method, use \( (p_0)^{n-1} \), \( (p_1)^{n} \), \( (p_2)^{n+1} \). Then you need to update them after every time iteration while convergence is not reached.

\[ i = 0, 1, \ldots, N \]
\[ j = 0, \ldots, M \]
\[ n = 1, 2, \ldots, N+1 \]

vii) a) Start computing \( (p_1)^{0} \); the rest of the values \( (p_1)^{0}_{ij} = 0 \) due to the IC’s. b) Also assume \( (p_0)^{n-1}_{ij} = 0 \) for all \( i, j \).

C) Now, you can start generating values for \( (p_2)^{1}_{ij} \), \( i \geq 1 \), \( j = 0, \ldots, M \).
Viii) Compute the total pressure:

\[ p = p_{sc} + P_{inc}, \]

And make a surface graph for it, in the whole domain \( r_0 < r < r_\infty, \quad 0 \leq \theta \leq 2\pi. \)

ix) Compare your numerical solution for \( p_{sc}(r,\theta,t) \)

against the exact solution discussed in class.

for a) \( r = 3, \quad \theta \in [0,2\pi]. \)

b) \( \theta = \frac{\pi}{4}, \quad 1 \leq r \leq r_\infty. \)

- Draw a graph for both in the same \( \theta_k \)-plane and \( \theta_{k+1} \)-plane, respectively.

- Also compute the error in maximum norm.

x) Compute the normalized differential cross section or far field \( S(r_\infty) \)

\[ 1 \frac{(\pi/2)}{A_{\theta}(\theta)} \mid \frac{d}{d\theta} \mid \frac{d}{d\theta} \]

from your numerical solution and also from your exact solution (as explained in class)

and compare both in the same graph \( \Theta_0 \)-plane.
xi) Improve the order of approximation on your radiation condition, replacing Sommerfeld radiation condition by

\[ \frac{\partial \psi_{sc}}{\partial t} + c \frac{\partial \psi_{sc}}{\partial r} + c \frac{\psi_{sc}}{2r} \]  \quad (\star)

Redo experiments: 2nd, 3rd and 4th column again with the new rad. cond.

b) Prove that (\star) is \( O(1/r^{3/2}) \) when \( r \to \infty \).

xii) Make comments about your results in the different experiments. I hope you will enjoy and learn a lot from this project.