Chapter 2 Exercises

From: *Finite Difference Methods for Ordinary and Partial Differential Equations*

Exercise 2.1  *(inverse matrix and Green’s functions)*

(a) Using the Green’s function, write out the $5 \times 5$ matrix $A$ from (2.43) for the boundary value problem $u''(x) = f(x)$ with $u(0) = u(1) = 0$ for $h = 0.25$.

(b) Write out the $5 \times 5$ inverse matrix $A^{-1}$ explicitly for this problem. Verify that the matrix that you obtain is indeed the inverse of the matrix $A$.

(c) If $f(x) = x$, determine the discrete approximation to the solution of the boundary value problem on this grid and sketch this solution and the five Green’s functions whose sum gives this solution.

Exercise 2.2  *(Green’s function with Neumann boundary conditions)*

(a) Determine the Green’s functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e, find the function $G(x, \bar{x})$ solving

$$u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0.$$  

Then, graph this Green’s function.

Also, find the functions $G_0(x)$ solving

$$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$  

and $G_1(x)$ solving

$$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$  

As a result, show that the solution $u$ of this Neumann problem can be represented by

$$u(x) = \alpha(x - 1) + \int_0^1 f(\bar{x})G(x; \bar{x}) \, d\bar{x} + \beta.$$  

(Ex2.2a)

(b) Using this as guidance, to find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the $5 \times 5$ matrices $A$ and $A^{-1}$ for the case $h = 0.25$. Verify that the matrix that you obtain is indeed the inverse of the matrix $A$.

(c) If $f(x) = x$, determine the discrete approximation to the solution of the boundary value problem on the grid obtained in the previous item and sketch this solution and the five Green’s functions whose sum gives this solution.
(d) Use the trapezoidal rule to approximate the integral in (Ex2.2a) on the grid points. Then, evaluate the resulting function of $x$ on the same grid points and show that the linear system $A^{-1}F = U$ obtained from this discrete approximation corresponds to the second order finite difference approximation of the original Neumann problem with centered difference approximation for the Neumann condition at $x = 0$. 