Fredholm Equations with Separable Kernels.

Separable Kernel: \( K(x, y) = \sum_{j=1}^{n} d_j(x) \beta_j(y) \) \hspace{1cm} (1)

Where \( \{d_j(x)\}_{j=1}^{n} \) are real valued continuous functions in \( C[a,b] \).

Linearly independent. The same is true for the set \( \{\beta_j(y)\}_{j=1}^{n} \).

Now, consider \( Ku - \lambda u = f \), \( \lambda \) real \( ^{(a)} \)

or

\[
\int_{a}^{b} [K(x, y)u(y)dy - \lambda u(x)] = f(x)
\]

Substituting (1) into (2), we obtain

\[
\int_{a}^{b} \left( \sum_{j=1}^{n} d_j(x) \beta_j(y) u(y)dy \right) - \lambda u(x) = f(x)
\]

which is equivalent to

\[
\sum_{j=1}^{n} [d_j(x) \int_{a}^{b} \beta_j(y)dy] u(y)dy - \lambda u(x) = f(x)
\]

(2.1)

(a) Remark: \( \lambda \) may be complex, but that would lead to complex solutions.
Using the definition of the inner product

\[ (f, g) = \int_a^b f(x)\overline{g(x)}\,dx \]

We can rewrite (2.1) as

\[ \sum_{j=1}^n \alpha_j(x) (u, \beta_j) - \lambda u(x) = f(x) \]

Calling \((u, \beta_j) = c_j\)

\[ \sum_{j=1}^n \alpha_j(x) c_j - \lambda u(x) = f(x). \] (3)

Solving for \(u(x)\) (in case \(\lambda \neq 0\)).

\[ u(x) = \frac{1}{\lambda} \left[ \sum_{j=1}^n \alpha_j(x) c_j - f(x) \right]. \] (4)

Thus, if somehow we could determine the set of constants \(c_j\)'s via (4), then the solution \(u(x)\) for the 2nd kind Fredholm integral equation (2) could be obtained from (4).
Key idea to determine $c_i$'s is to multiply equation (3) by $\beta_i(x)$ and $\int_a^b dx$, to transform (3) into an algebraic linear system.

\[
\int_a^b \left[ \sum_{j=1}^n \left( \int_a^b d_j(x) c_j \right) \beta_i(x) \right] dx - \lambda \int_a^b u(x) \beta_i(x) dx = \int_a^b f(x) \beta_i(x) dx
\]

or interchanging summation and integration.

\[
\sum_{j=1}^n \left( \int_a^b d_j(x) \beta_i(x) dx \right) c_j - \lambda c_i = f_i
\]

Where \( f_i = \int_a^b f(x) \beta_i(x) dx = (f, \beta_i) \).

or

\[
\sum_{j=1}^n (\beta_i, \beta_j) c_j - \lambda c_i = f_i
\]

(5)
Expanding these equations

\[ i=1 : (\beta_1, d_1) C_1 + (\beta_1, d_2) C_2 + \ldots + (\beta_1, d_n) C_n - \lambda C_1 = f_1 \]

\[ i=2 : (\beta_2, d_1) C_1 + (\beta_2, d_2) C_2 + \ldots + (\beta_2, d_n) C_n - \lambda C_2 = f_2 \]

\[ i=n : (\beta_n, d_1) C_1 + (\beta_n, d_2) C_2 + \ldots + (\beta_n, d_n) C_n - \lambda C_n = f_n \]

In matrix notation:

\[
\begin{pmatrix}
(\beta_1, d_1) & (\beta_1, d_2) & \ldots & (\beta_1, d_n) \\
(\beta_2, d_1) & (\beta_2, d_2) & \ldots & (\beta_2, d_n) \\
\vdots & \vdots & \ddots & \vdots \\
(\beta_n, d_1) & (\beta_n, d_2) & \ldots & (\beta_n, d_n)
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2 \\
\vdots \\
C_n
\end{pmatrix}
- 
\begin{pmatrix}
\lambda & 0 & \ldots & 0 \\
0 & \lambda & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2 \\
\vdots \\
C_n
\end{pmatrix}
= 
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_n
\end{pmatrix}
\]
or in matrix notation

\[
A \hat{C} - \lambda I \hat{C} = \tilde{f}
\]

where

\[
A = \left[ (\beta_i, x_j) \right], \quad \tilde{f} = \left( \begin{array}{c} f_1 \\ \vdots \\ f_n \end{array} \right)
\]

Equ. (6) can also be written as

\[
(A - \lambda I) \hat{C} = \tilde{f}.
\]

Recall that our purpose is to find \( \hat{C} \). So far, our original integral equation (2) has been reduced to the algebraic linear system of equation (7).

Therefore, if the vector \( \hat{C} = \left( \begin{array}{c} c_1 \\ \vdots \\ c_n \end{array} \right) \) satisfies (3) then it also satisfies (7). The reverse of this statement can also be proved.

Therefore, if \( \hat{C} \) is a solution of (7) and \( \lambda \neq 0 \), the vector \( \hat{C} \) can be used in eq. (4) to obtain a solution \( u(x) \) of our original Fredholm integral equation.
Back to equation (7): 
\[(A - \lambda I) \hat{C} = \hat{f}\]  
(7)

This algebraic linear system will have a unique solution iff 
\[\det(A - \lambda I) \neq 0.\]

In other words if and only if \(\lambda\) is not an eigenvalue of \(A\).

Also, this unique solution can be obtained as 
\[\hat{C} = (A - \lambda I)^{-1} \hat{f}.\]

This condition in turns implies that our original Fredholm integral equation also has a unique solution given by (4) 
\[U(x) = \frac{1}{\lambda} \left[ \sum_{j=1}^{n} d_j(x)C_j - f(x) \right].\]

Otherwise, if \(\det(A - \lambda I) = 0\), or equivalently, \(\lambda\) is an eigenvalue of \(A\).

Eqn. (7) will have infinitely many solutions if \(f\) is in the range of \((A - \lambda I)\).

If \(f\) is not a linear comb. of the column vectors of \(A - \lambda I\), there is no solution for (7). The same statements are true for the integral eqn. (2) with separable kernel.
This discussion is summarized in the following theorem:

Theorem (Fredholm Alternative).

For the integral equ. (2) with separable kernel defined by (1), \( \lambda \neq 0 \), and the matrix \( A \) defined by \( A = (a_{ij}(x)) \), the following statements are valid:

1) If \( \lambda \) is not an eigenvalue of \( A \) then (2) has a unique solution given by

\[
U(x) = \frac{1}{\lambda} \left[ -f(x) + \sum_{j=1}^{n} a_{j}(x) c_j \right], \quad c_j \in (u_{j}, b_{j})
\]

2) If \( \lambda \) is an eigenvalue of \( A \) then,

a) If \( f \) is in the range of \( (A - \lambda I) \), equ. (2) has infinitely many solutions.

b) If \( f \neq 0 \) is not in the range of \( (A - \lambda I) \) then equ. (2) has no solution.
Back to Fredholm Integral Equations.

So far, we have an algebraic nonhomogeneous linear system of equations:

\[ \sum_{j=1}^{n} (\beta_i, \alpha_j) c_j - \lambda c_i = (f, \beta_i) \quad (18.1) \]

or \[(A - \lambda I) \tilde{C} = \tilde{F} \]

where 
\[ A = (\alpha_i, \beta_j), \quad \tilde{C} = \begin{pmatrix} (u, \beta_1) \\ \vdots \\ (u, \beta_n) \end{pmatrix}, \quad \tilde{F} = \begin{pmatrix} (f, \beta_1) \\ \vdots \\ (f, \beta_n) \end{pmatrix} \]

which is related to the linear Fredholm integral equation with separable kernel

\[ Ku - \lambda u = f \quad (18.2) \]

where
\[ Ku = \int_{a}^{b} K(x, y) u(y) dy \]

and
\[ K(x, y) = \sum_{j=1}^{n} \alpha_j (x) \beta_j (y) \]

(18.2) written in full is
\[ \int_{a}^{b} K(x, y) u(y) dy - \lambda u(x) = f(x) \]
\[ \int_{a}^{b} \left( \sum_{j=1}^{n} d_j(x) \beta_j(y) \right) u(y) \, dy - \lambda u(x) = f(x). \]

\[ \Rightarrow \sum_{j=1}^{n} d_j(x) \int_{a}^{b} u(y) \beta_j(y) \, dy - \lambda u(x) = f(x). \]

\[ \Rightarrow \sum_{j=1}^{n} d_j(x) (u, \beta_j) - \lambda u(x) = f(x) \]

\[ \Rightarrow \left[ \sum_{j=1}^{n} d_j C_j - \lambda u(x) = f(x) \right] \quad (19.1) \]

from (19.1), if \( \lambda \neq 0 \),

\[ u(x) = \frac{1}{\lambda} \left[ \sum_{j=1}^{n} d_j C_j - f(x) \right] \quad (19.2) \]

So for \( \lambda \neq 0 \), the procedure to follow to obtain the
Solns. for the integral equ. (18.2) is

1) Solve the Algebraic System (18.1) to find \( \hat{C} = \left( \hat{c}_n \right) \)

2) If there is such a \( \hat{C} \) Soln. of (18.1)
Substitute it in (19.2) to obtain \( u(x) \).
Now, let's study the integral equation for $\lambda = 0$.

\[ Ku = f \]

K is separable kernel (20.1)

or

\[ \sum_{j=1}^{n} c_j \phi_j(x) = f(x). \]

There are two cases:

a) $Ku = 0$, when $f(x) = 0$. Homogeneous case.

b) $Ku = f$, $f(x) \neq 0$. Nonhomogeneous case.

Compare with $A\hat{x} = \hat{b}$, where $A$ non-neg matrix.

**Case (a)** Homogeneous case

$Ku = 0$ is equiv. to $\sum_{j=1}^{n} c_j \phi_j(x) = 0(x)$

\[ \text{multiplying by \text{bixx} and \text{C} & \text{dx}.} \]

Since $\phi_j$ is a liner indep set, we don't need to use it.

$\Rightarrow c_j = 0, \quad j = 1, ..., n$.

Therefore,

\[ (U_1, \beta_j) = c_j = 0, \quad j = 1, 2, ..., n. \]

So

\[ u(x) \perp \beta_j(x), \quad j = 1, ..., n. \]

Now, $u(x) \in L^2[a, b]$ or $C[a, b]$ both infinite-dimensional spaces. Therefore, there are infinitely many $u(x)$ solutions of (20.1)

linearly indep.

\[ \text{Rxn: } \beta = \mathbf{e}_1, \quad u_1 = \varepsilon_1, \quad u_2 = \varepsilon_2 \]

\[ \varepsilon_3 = \mathbf{e}_3 \]

Two lin indep. solns. only.

Now, think of $\mathbf{e}_1$ in an infinite-dim. space.
If Case (b) \( f(x) \neq 0 \), 
\[ Ku = f. \]

Nonhomogeneous case

\[ \sum_{j=1}^{n} c_j j(x) = f(x) \text{ is equiv to } Ku = f. \]

Clearly,

1. There is no solution if \( f(x) \) is not a linear combination of \( j(x)'s. \)

2. There are infinitely many solutions linearly indep.

If \( f(x) \) is a linear combination of \( \{ j(x) \}. \)

In fact, \( \sum_{j=1}^{n} c_j j(x) = f(x) = \sum_{j=1}^{n} f_j j(x). \)

\[ \text{lin.indep.} \quad C_j = f_j, \quad j = 1, 2, \ldots, n \Rightarrow (2, \beta_j) = f_j, \text{ for all}. \]

So any function \( u(x) \) in \( L_2[0,1] \) such that \( (2, \beta_j) = f_j \)

is a solution of \( (2.0.1) \). Because \( L_2[0,1] \) is infinite dim.

there are infinitely many linearly indep. solns. \( u(x) \) satisfying this property.

Proof: If \( \{ \beta_j \} \) were an orthonormal set, then \( (2, \beta_j) = f_j \)

means that the projection of \( u \) on \( \beta_j \) is \( f_j \).

In \( \mathbb{R}^3 \) for example, there are infinitely many \( u \) satisfying this property. However, they all lie in the same plane. Therefore there are only two linearly independent.
The previous discussion can be summarized in the next theorem.

**Theorem 1.3** Consider

\[ Ku = f, \]  

where the kernel of \( K \) is separable as defined previously, then

a) If \( f(x) \) is a linear combination of \( dj(x) \)'s, then there are infinitely many solutions linearly independent for \((2.1)\).

b) Otherwise, there is not solution.

This theorem includes the case \( Ku = 0 \).

Thus, for \( Ku = 0 \) there are always infinitely many solutions, which is different than \( Ax = 0 \) (a \( n \times n \) matrix).

Therefore, \( \lambda = 0 \) is always an eigenvalue of the Fredholm integral operator with separable kernel. Moreover, \( \lambda = 0 \) is always an eigenvalue of infinite multiplicity.
Another important result is the relationship between eigenvalues of

\[ Ku = \lambda u \]  

(23.1)

and the corresponding linear system

\[ A \vec{v} = \lambda \vec{v} \]  

(23.2)

\[ A = \left( \begin{array}{ccc} \beta_1 & \cdots & \beta_n \\ \vdots & \ddots & \vdots \\ \beta_n & \cdots & \beta_1 \end{array} \right) \]

Thus:

- The eigenvalues \( \lambda \neq 0 \) of (23.1) are the same eigenvalues of (23.2).
- The multiplicity of these eigenvalues is at most \( n \).
- \( \lambda = 0 \) is always an eigenvalue of \( K \) and its multiplicity is infinite.

Example 1.4 Consider the two Fredholm integral equations

\( a) \quad \int_0^1 (1-3x)u(y) \, dy - \lambda u(x) = \frac{x}{2} \quad (23.3) \)

\( b) \quad \int_0^1 (1-3x)u(y) \, dy - \lambda u(x) = 1-3x \quad (23.4) \)

i) Find their solutions if they exist for any \( \lambda \).
ii) Find the eigenvalues \( \lambda \neq 0 \) of \( Ku = \lambda u \), and find their corresponding eigenvectors.
Let's start working on (iii). We will obtain the eigenvalues from the associated linear system

\[(A - \lambda I) \vec{v} = 0\]

Where \[A = \begin{pmatrix} \beta_1 & d_1 \\ \beta_2 & d_2 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \]

In our case, \(d_1(x) = 1\), \(\beta_1(y) = 1\), \(d_2(x) = -3x\), \(\beta_2(y) = y\).

Therefore, \(\int_0^1 1 \, dx = 1\), \(\int_0^1 (3x) \, dx = -\frac{3}{2}\), \(\int_0^1 x \, dx = \frac{1}{2}\), \(\int_0^1 -3x^2 \, dx = -1\).

So, \[A = \begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \]

Eigenvalues: \(\lambda_1 = \frac{1}{2}\), \(\lambda_2 = -\frac{1}{2}\).

For \(\lambda \neq \pm \frac{1}{2}\), there is a unique soln for (23.3) and (23.4) given by

\[U(x) = \frac{1}{\lambda} \left[ -f(x) + \sum_{j=1}^{2} c_j \varphi_j(x) \right] \]
While the vector \( \vec{C} \) is the unique solution of

\[
(A - \lambda I) \vec{C} = \vec{f}
\]

(Case a): \[
f_1 = (\beta_0(\lambda x), f_0) = \int_0^1 (1) \frac{x}{2} \, dx = \frac{x^2}{4} \bigg|_0^1 = \frac{1}{4}
\]

\[
f_2 = (\beta_2(\lambda x), f_2) = \int_0^1 \frac{x}{3} \, dx = \frac{1}{6}
\]

Then, the linear system that we need to solve is

\[
\begin{pmatrix}
1 - \lambda & -3/2 \\
1/2 & 1 - \lambda
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
1/4 \\
1/6
\end{pmatrix}
\Rightarrow \vec{C} = \begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
-1/3 \\
1/6
\end{pmatrix}
\]

And

\[
U(x) = \frac{1}{\lambda = 1} \left[ -\frac{x}{2} + (1/3) \left( -\frac{1}{3} \right) \right] =
\]

\[
= -\frac{x}{2} + \frac{1}{3} \left( -\frac{1}{3} \right)
\]

or unique solution:

\[U(x) \equiv -\frac{1}{3}\]

Verification: Substituting in (23.3): \( \lambda = 1 \), \( u(0) = -\frac{1}{6} \)

\[
\int_0^1 (-\frac{1}{3}x) \, dx + \frac{1}{3} = -\frac{1}{3} \left( \frac{1}{3} \right) + \frac{1}{3} x \bigg|_0^1 = \frac{x^2}{3} \bigg|_0^1 = \frac{x}{3} \checkmark
\]
If $\lambda = \frac{1}{2}$ (for example @1),

the associated linear system is given by

$$(A - \frac{1}{2} I)c = f$$

or

$$\begin{pmatrix}
\frac{1}{2} & -\frac{3}{2} \\
\frac{1}{2} & -\frac{3}{2}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{4} \\
\frac{1}{6}
\end{pmatrix}
$$

there is

$$\begin{cases}
\frac{1}{2} c_1 - \frac{3}{2} c_2 = \frac{1}{4} \\
\frac{1}{2} c_1 - \frac{3}{2} c_2 = \frac{1}{6}
\end{cases} \Rightarrow \text{no solution}
$$

In case (b), the associated linear system is given by

$$\begin{pmatrix}
\frac{1}{2} & -\frac{3}{2} \\
\frac{1}{2} & -\frac{3}{2}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} =
\begin{pmatrix}
-\frac{1}{2} \\
-\frac{1}{2}
\end{pmatrix}
$$

Infinitely many solns:

$$\frac{1}{2} c_1 - \frac{3}{2} c_2 = -\frac{1}{2} \Rightarrow c_1 = -1 + 3c_2$$

$$\hat{c} = (\frac{-1}{3}) + c_2 (1)$$

If $c_2 = 1 \Rightarrow c_1 = 2 \Rightarrow \hat{c} = (2)$$

and the solution for the integral equation is given by

$\mathcal{U}(x) = \frac{1}{\frac{1}{2}} \left( 2 + (-3x) - 13x \right) = 2$

or

$$\boxed{\mathcal{U}(x) = 2}$$

is one possible solution.
ii) Finding eigenvectors:

\[ Ku = \lambda u \rightarrow \tilde{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ Corresponding eigenvector} \]

If \( \lambda = \frac{1}{2} \), we can use formula

\[ U(x) = \frac{1}{\lambda} \left[ \sum_{j=1}^{n} a_j(x) c_j \right] \]

\[ \Rightarrow U(x) = \frac{1}{\frac{1}{2}} \left( 3(1) + 1(-2x) \right) = 2 \left( 3 - 2x \right) = 6 - 6x \]

of eigenfunction corresponding to \( \lambda = \frac{1}{2} \)

\[ U(x) = 6(1-x) \]

Any multiple of \( \tilde{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \) is also an eigenvector corresponding to \( \lambda = \frac{1}{2} \). Therefore, any multiple of \( U(x) = 1-x \) is also an eigenfunction.

Similarly, for \( \lambda = -\frac{1}{2} \) corresponding eigenfunctions are multiples of \( U(x) = 1-3x \).